DIXMIER TRACE FOR TOEPLITZ OPERATORS ON SYMMETRIC DOMAINS

HARALD UPMEIER AND KAI WANG

ABSTRACT. For Toeplitz operators on bounded symmetric domains of arbitrary rank, we define a Hilbert quotient module corresponding to partitions of length 1 and prove that it belongs to the Macaev class $\mathcal{L}^{n,\infty}$. We next obtain an explicit formula for the Dixmier trace of Toeplitz commutators in terms of the underlying boundary geometry.

0. INTRODUCTION

The Dixmier trace of Hilbert space operators [6], of fundamental importance for pseudo-differential operators [5, 28], has recently found deep applications in **complex analysis**, for Hankel and Toeplitz operators on strictly pseudo-convex domains [1, 10, 11, 12, 18] and for homogeneous Hilbert quotient modules over the unit ball [8, 9, 15, 16]. In these applications the underlying operators are essentially normal, i.e. commutators are compact; more precisely, belong to certain norm ideals of Schatten type.

In this paper we are concerned with operators of Toeplitz or Hankel type which are not essentially commuting. These operators arise naturally when the underlying domain $D \subset \mathbf{C}^d$ is not strictly pseudo-convex or does not have a smooth boundary. The most important case is the so-called **hermitian bounded symmetric domains** $D = G/K$ of arbitrary rank r, which generalize the unit disk and the unit ball of rank 1. In this paper, we construct a suitable Hilbert quotient module of the Hardy space over the Shilov boundary S and study the associated 'sub-Toeplitz' operators. Our first main result shows that commutators of such operators belong to the Macaev class $\mathcal{L}^{n,\infty}$, for a suitable n related to the geometry of D. The second main result is an explicit formula for the Dixmier trace of products of such operators, in terms of a Jordan theoretic Grassmann-type manifold.

The results of this paper can be generalized to cover the weighted Bergman spaces instead of the Hardy space, at least for the continuous part of the Wallach set [14]. On the other hand, extending these results to all smooth functions $f \in \mathcal{C}^\infty(S)$ will be more challenging, even for the basic case of rank 2-domains (involving pseudo-differential operators on spheres [3, 4]). Finally, the higher strata of the boundary of D give rise to a family of smooth extensions [27] and it is of interest to develop a family version of the Dixmier trace (involving cyclic cohomology) for the associated Toeplitz commutators.

1. SYMMETRIC DOMAINS AND TOEPLITZ OPERATORS

Let D be an irreducible bounded symmetric domain of rank r in a complex vector space Z of finite dimension d. The unit ball $D = \mathbf{B}_d \subset \mathbf{C}^d$ corresponds to rank $r = 1$. Denote by $G = Aut(D)$ the biholomorphic automorphism group, and put

$$K := \{g \in G : g(0) = 0\}.$$

Then $D = G/K$. It is well known [14, 19] that D can be realized as the open unit ball of an irreducible *hermitian Jordan triple* Z. Thus Z is a complex vector space endowed with a Jordan triple product

$$u, v, w \mapsto \{uv^*w\} \in Z \qquad \forall\, u, v, w \in Z.$$

2010 *Mathematics Subject Classification.* 32M15; 42B35; 47B35.

Key words and phrases. bounded symmetric domain, Toeplitz operator, Dixmier trace.

The second author was partially supported by NSFC (11271075,11420101001), the Alexander von Humboldt Foundation and Laboratory of Mathematics for Nonlinear Science at Fudan University.

Then $K = Aut(Z)$ is the linear group of all triple automorphisms of Z. Let S be the Shilov boundary of D. Since K acts transitively on S, there exists a unique $K-$invariant probability measure ds on S. Denote by $L^2(S)$ the space of L^2-integrable functions, with inner product

$$(f|g)_S := \int\limits_S ds\, \overline{f(s)}\, g(s), \tag{1.1}$$

and define the **Hardy space**

$$H^2(S) = \{\psi \in L^2(S) : \psi \text{ holomorphic on } D\}.$$

For a bounded function f, define the **Toeplitz operator**

$$T_f \psi = P_{H^2(S)}(f\psi) \qquad \forall \psi \in H^2(S).$$

In previous work [23, 24] it was shown that Toeplitz operators T_f with smooth symbol function $f \in \mathcal{C}^\infty(S)$, acting on $H^2(S)$, generate a C^*-algebra $\mathcal{T}(S)$ which is not essentially commutative (if $r > 1$) but has a **composition series**

$$\mathcal{K} = \mathcal{I}_1 \subset \mathcal{I}_2 \subset \cdots \subset \mathcal{I}_r \subset \mathcal{T}(S) = \mathcal{I}_{r+1},$$

starting with the compact operators $\mathcal{K}$, such that the subquotients $\mathcal{I}_{k+1}/\mathcal{I}_k$ are essentially commutative. More precisely, there is a stable isomorphism

$$\mathcal{I}_{k+1}/\mathcal{I}_k \approx \mathcal{C}(S_k) \otimes \mathcal{K},$$

where S_k denotes the K-homogeneous manifold of all 'tripotents' of rank k. (Similar results hold for Toeplitz operators on weighted Bergman spaces over D, as shown in [26].) An element $c \in Z$ such that $\{cc^*c\} = c$ is called a **tripotent**. Every tripotent induces a **Peirce decomposition**

$$Z = Z_c^2 \oplus Z_c^1 \oplus Z_c^0,$$

where $Z_c^\alpha := \{z \in Z : \{cc^*z\} = 2\alpha z\}$. The Peirce 2-space is a Jordan $*$-algebra with unit element c and involution $z \mapsto \{cz^*c\}$. The self-adjoint part $X_c \subset Z_c^2$ is a so-called **euclidean Jordan algebra** [14]. Let $(z|w)$ denote the K-invariant inner product normalized by the condition $(c|c) = 1$ for each minimal tripotent $c \in Z$. Let $\mathcal{P}(Z)$ be the algebra of all (holomorphic) polynomials on Z, endowed with the K-invariant **Fischer-Fock inner product**

$$(p|q)_Z := \frac{1}{\pi^d} \int\limits_Z dz\, e^{-(z|z)} \overline{p(z)}\, q(z) \tag{1.2}$$

for all $p, q \in \mathcal{P}(Z)$. By [14, 25] the natural action of K on $\mathcal{P}(Z)$ induces a multiplicity-free **Peter-Weyl decomposition**

$$\mathcal{P}(Z) = \sum_\lambda \mathcal{P}_\lambda(Z), \tag{1.3}$$

where

$$\lambda = \lambda_1 \geq \ldots \geq \lambda_r \geq 0$$

runs over all integer **partitions** of length $\leq r$. The decomposition (1.3) is orthogonal under (1.2). We let $\mathbf{N}_+^r$ denote the set of all such partitions. As usual we will identify partitions that differ only by zeros. Then

$$\mathbf{N}_+^r = \bigcup_{\ell=1}^r \mathbf{N}_+^\ell,$$

where $\mathbf{N}_+^\ell = \{\lambda \in \mathbf{N}_+^r : \lambda_{\ell+1} = 0\}$. As a special type of partition we denote

$$k_\ell := (k, \ldots, k, 0 \ldots, 0)$$

for $1 \leq \ell \leq r$ and $k \in \mathbf{N}$ repeated ℓ times. Choose a frame $e_1, \ldots, e_r$ of minimal tripotents. The associated joint Peirce decomposition [19] defines two numerical invariants a, b for Z such that

$$\rho := \frac{d}{r} = 1 + \frac{a}{2}(r-1) + b.$$

For the Hilbert unit ball ($r = 1$) we put $a = 2$ and $b = d - 1$. Thus $b = 0$ only for the unit disk. In case $b = 0$ the Jordan triple Z is actually a **Jordan algebra** with unit element

$$e := e_1 + \cdots + e_r.$$

In this case Z carries a **Jordan determinant** $N = N_r$ which is normalized by $N(e) = 1$. For $1 \leq \ell \leq r$ denote by N_ℓ the Jordan determinant polynomial for the Peirce 2-space $Z^2_{e_1+\ldots+e_\ell}$. As shown in [25] $\mathcal{P}_\lambda(Z)$ has the highest weight vector

$$N_\lambda(z) := N_1(z)^{\lambda_1-\lambda_2} N_2(z)^{\lambda_2-\lambda_3} \cdots N_r(z)^{\lambda_r}. \tag{1.4}$$

The **multi-variable Pochhammer symbol** is the product

$$(s)_\lambda := \prod_{i=1}^{r} (s - \frac{a}{2}(i-1))_{\lambda_i}$$

of the usual Pochhammer symbols $(\nu)_m = \prod_{i=1}^{m} (\nu + i - 1)$. By [23, 14] the inner products (1.2) and (1.2) are related by

$$(p|q)_S = \frac{1}{(\rho)_\lambda} (p|q)_Z, \qquad \forall\, p, q \in \mathcal{P}_\lambda(Z). \tag{1.5}$$

We note the relation

$$\frac{(\rho)_\lambda}{(\rho - b)_\lambda} = \prod_{j=1}^{r} \frac{(\lambda_j + 1 + \frac{a}{2}(r-j))_b}{(1 + \frac{a}{2}(r-j))_b}.$$

Proposition 1.1. *For $\lambda \in \mathbf{N}^r_+$ we have*

$$\|N_\lambda\|^2_S = \frac{(\rho - b)_\lambda}{(\rho)_\lambda} \prod_{1 \leq i < j \leq r} \frac{(1 + \frac{a}{2}(j-i-1))_{\lambda_i - \lambda_j}}{(1 + \frac{a}{2}(j-i))_{\lambda_i - \lambda_j}}. \tag{1.6}$$

Proof. Using the reciprocity relation

$$\frac{(x+b)_m}{(x)_m} = \frac{(x+m)_b}{(x)_b} \tag{1.7}$$

for integers $0 \leq b \leq m$, the assertion follows from [23] or (for tube domains) [14, Proposition XI.4.3]. ☐

For any partition λ let

$$P_\lambda : \mathcal{P}(Z) \to \mathcal{P}_\lambda(Z) \tag{1.8}$$

denote the orthogonal projection. If $f_u(z) = (z|u)$ is a linear functional associated with $u \in Z$, we simply write $T_u := T_{f_u}$. Moreover, u^∂ denotes the directional derivative. By [23, Theorem 2.11] we have

$$T^*_u q = \sum_i P_{\lambda-[i]} T^*_u q = \sum_{i=1}^{r} \frac{1}{\lambda_i + \frac{a}{2}(r-i) + b} P_{\lambda-[i]} u^\partial q \tag{1.9}$$

for all $q \in \mathcal{P}_\lambda(Z)$, where

$$[i] = (0, \ldots, 0, 1, 0, \ldots, 0)$$

with 1 at position i. More precisely, only those terms occur where $\lambda - [i]$ is again a partition.

Definition 1.2. *Let $\mathcal{S}$ denote the set of all sequences*

$$c_m = c_0 + \frac{c_1}{m+1} + \mathfrak{o}_m,$$

where c_0, c_1 are constants and the sequence $\{m^2 \mathfrak{o}_m\}_m$ is bounded. Let $\mathcal{S}_+$ denote the set of sequences in $\mathcal{S}$ with $c_0 > 0$.

It is clear that $\mathcal{S}$ is closed under taking finite sums and products of sequences. $\mathcal{S}_+$ is also closed under taking quotients.

Lemma 1.3. *Let $\alpha, \gamma \in \mathbf{N}^{r-1}_+$. Then $\left\{ \frac{\|N_{m-k,\gamma}\|^2_S}{\|N_{m,\alpha}\|^2_S} \right\}_m \in \mathcal{S}_+$.*

Proof. In terms of the falling Pochhammer symbol $(m)_j^* = \prod_{i=1}^{j}(m+1-i)$, (1.6) implies

$$\frac{\|N_{m-k,\gamma}\|_S^2}{\|N_{m,\alpha}\|_S^2} = C\prod_{j=1}^{r-1}\frac{(m-\alpha_j+\frac{a}{2}j)^*_{k+\gamma_j-\alpha_j}}{(m-\alpha_j+\frac{a}{2}(j-1))^*_{k+\gamma_j-\alpha_j}}$$

whenever $k+\gamma_j \geq \alpha_j$. Since each factor belongs to $\mathcal{S}_+$, the assertion follows. □

Lemma 1.4. *Let $\ell \leq r$ and $\lambda \in \mathbf{N}_+^\ell$. Then*

$$T^*_{N_\ell^k}N_\lambda = \prod_{j=1}^{\ell}\frac{(\lambda_j+\frac{a}{2}(\ell-j))^*_k}{(\lambda_j+\frac{a}{2}(r-j)+b)^*_k}N_{\lambda-k_\ell}, \qquad \forall\, k \leq \lambda_\ell.$$

Proof. Consider the Peirce 2-space $\tilde{Z} = Z^2_{e_1+\ldots+e_\ell}$ of rank ℓ and put $\tilde{\rho} = 1+\frac{a}{2}(\ell-1)$. Using $N_\lambda = N_\ell^k\, N_{\lambda-k_\ell}$ and applying (1.5) to $S \subset Z$ and $\tilde{S} \subset \tilde{Z}$, we obtain for $\phi \in \mathcal{P}(\tilde{Z})$

$$(\rho)_\lambda(\phi|T^*_{N_\ell^k}N_\lambda)_S = (\rho)_\lambda(N_\ell^k\phi|N_\lambda)_S = (N_\ell^k\phi|N_\lambda)_Z = (N_\ell^k\phi|N_\lambda)_{\tilde{Z}} = (\tilde{\rho})_\lambda(N_\ell^k\phi|N_\lambda)_{\tilde{S}}$$

$$= (\tilde{\rho})_\lambda(\phi|N_{\lambda-k_\ell})_{\tilde{S}} = \frac{(\tilde{\rho})_\lambda}{(\tilde{\rho})_{\lambda-k_\ell}}(\phi|N_{\lambda-k_\ell})_{\tilde{Z}} = \frac{(\tilde{\rho})_\lambda}{(\tilde{\rho})_{\lambda-k_\ell}}(\phi|N_{\lambda-k_\ell})_Z = \frac{(\tilde{\rho})_\lambda(\rho)_{\lambda-k_\ell}}{(\tilde{\rho})_{\lambda-k_\ell}}(\phi|N_{\lambda-k_\ell})_S.$$

Since ϕ is arbitrary, it follows that

$$T^*_{N_\ell^k}N_\lambda = \frac{(\tilde{\rho})_\lambda(\rho)_{\lambda-k_\ell}}{(\tilde{\rho})_{\lambda-k_\ell}(\rho)_\lambda}N_{\lambda-k_\ell}.$$

We have

$$\frac{(\rho)_\lambda}{(\rho)_{\lambda-k_\ell}} = \prod_{j=1}^{\ell}(\lambda_j+\frac{a}{2}(r-j)+b)^*_k. \tag{1.10}$$

Applying (1.10) to Z and $\tilde{Z}$, the assertion follows. □

We will now consider partitions $(m,0,\ldots,0) = m$ of length 1, with projection $P_m : H^2(S) \to \mathcal{P}_m(Z)$. Here $\mathcal{P}_m(Z)$ is spanned by the K-orbit of the conical polynomial N_1^m. As shown in [24], the projection

$$P := \sum_m P_m \tag{1.11}$$

on $H^2(S)$ belongs to the Toeplitz C^*-algebra $\mathcal{T}(S)$. For a partition λ choose an orthonormal basis $p_i \in \mathcal{P}_\lambda(Z)$, for the inner product (1.2). Then

$$A^\lambda := \sum_i T_{p_i}PT^*_{p_i}$$

is a K-invariant operator, independent of the choice of orthonormal basis. Since the decomposition (1.3) is multiplicity-free, every K-invariant operator T on $\mathcal{P}(Z)$ (or $H^2(S)$) is a 'diagonal' operator. Define

$$\lambda' := (\lambda_2,\ldots,\lambda_r) \in \mathbf{N}_+^{r-1}.$$

Lemma 1.5. *Let $p \in \mathcal{P}_\lambda(Z)$ such that $PT_p^*N_{m,\beta} \neq 0$. Then $\beta \leq \lambda \leq (m,\beta)$.*

Proof. For $\phi \in \mathcal{P}_{n,0}(Z)$ the non-zero components of $p\phi$ correspond to signatures μ obtained from λ by adding a horizontal n-strip [22, Proposition 5.3]. Thus

$$\mu' \leq \lambda \leq \mu.$$

It follows that $(\phi|T_p^*N_{m,\beta})_S = (p\,\phi|N_{m,\beta})_S$ is non-zero only if $\mu = (m,\beta)$ satisfies the above condition, which leads to $\beta \leq \lambda \leq (m,\beta)$. □

Since

$$\mathrm{Ran}(T_{p_i}P) \subset \sum_{\mu' \le \lambda \le \mu} P_\mu$$

by Lemma (1.5), it follows that

$$A^\lambda = \sum_{\mu' \le \lambda \le \mu} \frac{(N_\mu | A^\lambda N_\mu)_S}{\|N_\mu\|_S^2} P_\mu, \tag{1.12}$$

where

$$\frac{(N_\mu | A^\lambda N_\mu)_S}{\|N_\mu\|_S^2} = \frac{1}{\|N_\mu\|_S^2}(N_\mu | \sum_i T_{p_i} P T_{p_i}^* \, N_\mu)_S = \frac{1}{\|N_\mu\|_S^2} \sum_i \|PT_{p_i}^* N_\mu\|_S^2. \tag{1.13}$$

Proposition 1.6. *Let $\lambda \in \mathbf{N}_+^r$. Then $\left\{\frac{(N_{m,\lambda'}|A^\lambda N_{m,\lambda'})_S}{\|N_{m,\lambda'}\|_S^2}\right\}_m \in \mathcal{S}_+$.*

Proof. The proof is by induction on the length $\ell \le r$ of λ. Put $k := \lambda_\ell > \lambda_{\ell+1} = 0$. Then $\gamma := \lambda - k_\ell$ has length $< \ell$. Consider the Peirce 2-space $\tilde{Z} := Z^2_{e_1+\ldots+e_\ell}$ of rank ℓ. We may assume that a subfamily $p_i : i \in \tilde{I}$ is an orthonormal basis of $\mathcal{P}_\lambda(\tilde{Z})$. Since $\mathcal{P}_\lambda(\tilde{Z}) = N_\ell^k \, \mathcal{P}_\gamma(\tilde{Z})$, there exists a constant $c > 0$ such that $p_i = c \cdot N_\ell^k \, q_i$ for all $i \in \tilde{I}$, where $q_i \in \mathcal{P}_\gamma(\tilde{Z})$ is an orthonormal basis. For $m \ge \lambda_2$ it follows from Lemma (1.4) that $T^*_{N_\ell^k} N_{m,\lambda'} = c_m N_{m-k,\gamma'}$, where

$$c_m = \frac{(m + \frac{a}{2}(\ell-1))_k^*}{(m+b+\frac{a}{2}(r-1))_k^*} \prod_{j=2}^{\ell} \frac{(\lambda_j + \frac{a}{2}(\ell-j))_k^*}{(\lambda_j + b + \frac{a}{2}(r-1))_k^*}$$

belongs to $\mathcal{S}_+$, in view of the identity

$$\frac{m+a}{m+b} = 1 + \frac{a-b}{m} - \frac{(a-b)b}{m(m+b)}.$$

For $i \in I \setminus \tilde{I}$ we have $T^*_{p_i} N_{m,\lambda'} = 0$ since p_i belongs to the ideal generated by $\tilde{Z}^\perp$. It follows that

$$(N_{m,\lambda'}|A^\lambda N_{m,\lambda'})_S = \sum_{i \in I} (T^*_{p_i} N_{m,\lambda'} | PT^*_{p_i} N_{m,\lambda'})_S = \sum_{i \in \tilde{I}} (T^*_{p_i} N_{m,\lambda'} | PT^*_{p_i} N_{m,\lambda'})_S$$

$$= c^2 \sum_{i \in \tilde{I}} (T^*_{q_i} T^*_{N_\ell^k} N_{m,\lambda'} | PT^*_{q_i} T^*_{N_\ell^k} N_{m,\lambda'})_S = c^2 \cdot c_m^2 \sum_{i \in \tilde{I}} (T^*_{q_i} N_{m-k,\gamma'} | PT^*_{q_i} N_{m-k,\gamma'})_S.$$

Now consider the K-invariant operator

$$A^\gamma = \sum_{j \in J} T_{q_j} P T^*_{q_j},$$

where q_j, $j \in J$ is an orthonormal basis of $\mathcal{P}_\gamma(Z)$. We may assume that $q_i, i \in \tilde{I}$ are a subfamily of J. As above, we have $T^*_{q_j} N_{m-k,\gamma'} = 0$ whenever $j \in J \setminus \tilde{I}$. Therefore

$$A^\gamma N_{m-k,\gamma'} = \sum_{j \in J} T_{q_j} P T^*_{q_j} N_{m-k,\gamma'} = \sum_{i \in \tilde{I}} T_{q_i} P T^*_{q_i} N_{m-k,\gamma'}$$

and hence $(N_{m,\lambda'}|A^\lambda N_{m,\lambda'})_S = c^2 \cdot c_m^2 (N_{m-k,\gamma'}|A^\gamma N_{m-k,\gamma'})_S$. Since γ has length $< \ell$, the induction hypothesis implies that $\left\{\frac{(N_{m-k,\gamma'}|A^\gamma N_{m-k,\gamma'})_S}{\|N_{m-k,\gamma'}\|_S^2}\right\}_m \in \mathcal{S}_+$. It follows that the sequence

$$\frac{(N_{m,\lambda'}|A^\lambda N_{m,\lambda'})_S}{\|N_{m,\lambda'}\|_S^2} = c^2 \, c_m^2 \, \frac{(N_{m-k,\gamma'}|A^\gamma N_{m-k,\gamma'})_S}{\|N_{m-k,\gamma'}\|_S^2} \, \frac{\|N_{m-k,\gamma'}\|_S^2}{\|N_{m,\lambda'}\|_S^2}$$

belongs to $\mathcal{S}_+$, since Lemma (1.3) implies that $\left\{\frac{\|N_{m-k,\gamma'}\|_S^2}{\|N_{m,\lambda'}\|_S^2}\right\}_m \in \mathcal{S}_+$. $\square$

2. Hilbert submodule and sub-Toeplitz operators

The Hilbert sum
$$H_1^2(S)=\sum_m \mathcal{P}_m(Z)=\mathrm{Ran}(P)$$
will be called the **sub-Hardy space**. For smooth symbols $f\in\mathcal{C}^\infty(S)$ define the **sub-Toeplitz operator**
$$S_f:=P\,f\,P=P\,T_f\,P$$
as a bounded operator on $H_1^2(S)$. Let $\mathcal{A}$ be the $*$-algebra generated by S_p for polynomial symbols $p\in\mathcal{P}(Z)$. For $p,q\in\mathcal{P}(Z)$ we have
$$S_pS_q=S_{pq}$$
since $PT_qP^\perp=0$. Thus it often suffices to consider linear symbols $z\mapsto(z|u)$ for some $u\in Z$. We denote by S_u the corresponding operators.

Theorem 2.1. *For $\mu\in\mathbf{N}_+^r$ let $p\in\mathcal{P}(Z)$ satisfy $\deg p\le|\mu'|$. Then*
$$PT_p^*T_qP=S_{T_p^*q}\qquad\forall\,q\in\mathcal{P}_\mu(Z).$$

The proof is based on the following Lemma.

Lemma 2.2. *Let μ be a partition and $q\in\mathcal{P}_\mu(Z)$. Then we have for $u\in Z$ and each $h\in\mathcal{P}_{n,0}(Z)$,*
$$P_{\mu+n[1]-[i]}u^\partial P_{\mu+n[1]}hq=P_{\mu+n[1]-[i]}hP_{\mu-[i]}u^\partial q\qquad\forall\,i>1.$$

Proof. Write $hq=\sum\limits_\lambda P_\lambda hq$. The partitions λ occurring here satisfy $\lambda\ge\mu$ and hence $\lambda'\ge\mu'$. For such λ we have
$$u^\partial P_\lambda hq=\sum_j P_{\lambda-[j]}u^\partial P_\lambda hq.$$
Now assume $\lambda-[j]=\mu+n[1]-[i]$. If $j=1$ than $\lambda'=\mu'-[i]\not\ge\mu'$. Hence $j>1$. If $j\ne i$ then $\lambda'=\mu'-[i]+[j]\not\ge\mu'$. Hence $i=j$ and therefore $\lambda=\mu+n[1]$. This argument shows
$$P_{\mu+n[1]-[i]}u^\partial(hq)=\sum_\lambda P_{\mu+n[1]-[i]}u^\partial P_\lambda hq=P_{\mu+n[1]-[i]}u^\partial P_{\mu+n[1]}hq.\tag{2.1}$$
Since $u^\partial(hq)=q(u^\partial h)+h(u^\partial q)$ and $q(u^\partial h)$ has only components $\lambda\ge\mu$ which satisfy $\lambda'\ge\mu'$ it follows that
$$P_{\mu+n[1]-[i]}u^\partial(hq)=P_{\mu+n[1]-[i]}h(u^\partial q).\tag{2.2}$$
We next show that
$$P_{\mu+n[1]-[i]}h(u^\partial q)=\sum_j P_{\mu+n[1]-[i]}hP_{\mu-[j]}u^\partial q=P_{\mu+n[1]-[i]}hP_{\mu-[i]}u^\partial q.\tag{2.3}$$
In fact, since $hP_{\mu-[1]}u^\partial q$ cannot have a component λ with $\lambda_1=\mu_1+n$ we may assume $j>1$. If $j\ne i$, then the components $\lambda\ge\mu-[j]$ occurring in $hP_{\mu-[j]}u^\partial q$ satisfy $\lambda'\ge\mu'-[j]$ which implies $\lambda'\ne\mu'-[i]$. Thus (2.3) holds. Combining equations (2.1),(2.2) and (2.3), the assertion follows. □

Proof of Theorem (2.1)**.** We may assume that $p(z)=(z|u_1)\cdots(z|u_k)$. Let $\lambda=(\lambda_1,\lambda')$ be a partition such that $|\lambda'|\ge k$. Putting $[i_1,\ldots,i_k]=[i_1]+\ldots+[i_k]$ it follows from (1.9) that
$$T_{u_k}^*\ldots T_{u_1}^*\psi=\sum_{i_1,\ldots,i_k}P_{\lambda-[i_1,\ldots,i_k]}T_{u_k}^*\ldots P_{\lambda-[i_1]}T_{u_1}^*\psi$$
for all $\psi\in\mathcal{P}_\lambda(Z)$. If any $i_j=1$ then $(\lambda-[i_1,\ldots,i_k])'\ne0$. Therefore $(\lambda-[i_1,\ldots,i_k])'=0$ implies that all $i_j>1$. It follows that
$$PT_{u_k}^*\ldots T_{u_1}^*\psi=P\sum_{i_1>1,\ldots,i_k>1}P_{\lambda-[i_1,\ldots,i_k]}T_{u_k}^*\ldots P_{\lambda-[i_1]}T_{u_1}^*\psi.\tag{2.4}$$

Moreover, if $|\lambda'| > k$ we have $PT^*_{u_k}\dots T^*_{u_1}\psi = 0$. The same argument shows

$$Pu^\partial_k \dots u^\partial_1\psi = P\sum_{i_1>1,\dots,i_k>1} P_{\lambda-[i_1,\dots,i_k]}u^\partial_k\dots P_{\lambda-[i_1]}u^\partial_1\psi \tag{2.5}$$

and $|\lambda'| > k$ implies $Pu^\partial_k\dots u^\partial_1\psi = 0$. By Lemma (2.2) we have for $h \in \mathcal{P}_{n,0}(Z)$

$$P_{\mu+n[1]-[i_1]}u^\partial_1 P_{\mu+n[1]}hq = P_{\mu+n[1]-[i_1]}hP_{\mu-[i_1]}u^\partial_1 q.$$

Applying Lemma (2.2) to $P_{\mu-[i_1]}u^\partial_1 q$, we obtain

$$\begin{aligned} P_{\mu+n[1]-[i_1,i_2]}u^\partial_2 P_{\mu+n[1]-[i_1]}u^\partial_1 P_{\mu+n[1]}hq &= P_{\mu+n[1]-[i_1,i_2]}u^\partial_2 P_{\mu+n[1]-[i_1]}hP_{\mu-[i_1]}u^\partial_1 q\\ &= P_{\mu+n[1]-[i_1,i_2]}hP_{\mu-[i_1,i_2]}u^\partial_2 P_{\mu-[i_1]}u^\partial_1 q. \end{aligned}$$

More generally,

$$P_{\mu+n[1]-[i_1,\dots,i_k]}u^\partial_k\dots P_{\mu+n[1]-[i_1]}u^\partial_1 P_{\mu+n[1]}hq = P_{\mu+n[1]-[i_1,\dots,i_k]}hP_{\mu-[i_1,\dots,i_k]}u^\partial_k\dots P_{\mu-[i_1]}u^\partial_1 q. \tag{2.6}$$

Consider

$$Ph(T^*_{u_k}\dots T^*_{u_1}q) = Ph\sum_{i_1,\dots,i_k} P_{\mu-[i_1,\dots,i_k]}T^*_{u_k}\dots P_{\mu-[i_1]}T^*_{u_1}q = P\sum_{i_1,\dots,i_k}\sum_\lambda P_\lambda hP_{\mu-[i_1,\dots,i_k]}T^*_{u_k}\dots P_{\mu-[i_1]}T^*_{u_1}q.$$

Note the components $\lambda = (m,0)$ occurring here satisfy $\lambda' = 0 \geq (\mu-[i_1,\dots,i_k])'$. Since $|\mu'| \geq k$ this implies that all $i_j > 1$. Moreover, $m = |\lambda| = n + |\mu-[i_1,\dots,i_k]| = n+\mu_1+|\mu'-[i_1,\dots,i_k]| = n+\mu_1$. Therefore $\lambda = (n+\mu_1, 0)$ and hence

$$Ph(T^*_{u_k}\dots T^*_{u_1}q) = P\sum_{i_1>1,\dots,i_k>1} P_{\mu+n[1]-[i_1,\dots,i_k]}hP_{\mu-[i_1,\dots,i_k]}T^*_{u_k}\dots P_{\mu-[i_1]}T^*_{u_1}q. \tag{2.7}$$

We have $hq = \sum\limits_\lambda P_\lambda hq$, where $\lambda \geq \mu$ and the skew-partition $\lambda - \mu$ is a horizontal n-strip. Since $\lambda' \geq \mu'$ satisfies $|\lambda'| \geq |\mu'| \geq k$ the condition $(\lambda-[i_1,\dots,i_k])' = 0$ implies that all $i_j > 1$ and in addition all terms with $|\lambda'| > k$ vanish. Assuming $|\lambda'| = k$ it follows that $\lambda' = \mu'$ and hence $\lambda = \mu + n[1]$. This shows

$$PT^*_{u_k}\dots T^*_{u_1}(hq) = P\sum_\lambda T^*_{u_k}\dots T^*_{u_1}P_\lambda hq = PT^*_{u_k}\dots T^*_{u_1}P_{\mu+n[1]}hq.$$

With (2.4),(2.5), (2.6) and (2.7), we obtain

$$\begin{aligned} PT^*_{u_k}\dots T^*_{u_1}(hq) &= P\sum_{i_1>1,\dots,i_k>1} P_{\mu+n[1]-[i_1,\dots,i_k]}T^*_{u_k}\dots P_{\mu+n[1]-[i_1]}T^*_{u_1}P_{\mu+n[1]}hq\\ &= P\sum_{i_1>1,\dots,i_k>1}\frac{P_{\mu+n[1]-[i_1,\dots,i_k]}u^\partial_k\dots P_{\mu+n[1]-[i_1]}u^\partial_1 P_{\mu+n[1]}hq}{\big((\mu-[i_1,\dots,i_{k-1}])_{i_k}+\frac{a}{2}(r-i_k)+b\big)\dots\big(\mu_{i_1}+\frac{a}{2}(r-i_1)+b\big)}\\ &= P\sum_{i_1>1,\dots,i_k>1}\frac{P_{\mu+n[1]-[i_1,\dots,i_k]}hP_{\mu-[i_1,\dots,i_k]}u^\partial_k\dots P_{\mu-[i_1]}u^\partial_1 q}{\big((\mu-[i_1,\dots,i_{k-1}])_{i_k}+\frac{a}{2}(r-i_k)+b\big)\dots\big(\mu_{i_1}+\frac{a}{2}(r-i_1)+b\big)}\\ &= P\sum_{i_1>1,\dots,i_k>1} P_{\mu+n[1]-[i_1,\dots,i_k]}hP_{\mu-[i_1,\dots,i_k]}T^*_{u_k}\dots P_{\mu-[i_1]}T^*_{u_1}q = Ph(T^*_{u_k}\dots T^*_{u_1}q) \end{aligned}$$

It follows that $PT^*_p(hq) = Ph(T^*_p q)$. Since $h \in \mathcal{P}_{n,0}(Z)$ is arbitrary, $PT^*_pT_qP = PT_{T^*_p q}P = S_{T^*_p q}$. □

Applying Theorem (2.1) we obtain

Corollary 2.3. *If* $\deg p, \deg q \leq |\lambda'|$, *then* $PT^*_pA^\lambda T_qP \in \mathcal{A}$.

For $\beta \in \mathbf{N}^{r-1}_+$, consider the projections

$$P^\beta := \sum_{m\geq\beta_1} P_{m,\beta}.$$

Then $P^0 = P$.

Definition 2.4. *Define a diagonal operator* Λ *on* $\mathcal{P}(Z)$ *by*

$$\Lambda p_\lambda := \lambda_1\, p_\lambda, \qquad \forall\, p_\lambda \in \mathcal{P}_\lambda. \tag{2.8}$$

Let $Q_j := \oplus_{\lambda_1=j}\mathcal{P}_\lambda(Z)$ be the eigenvector subspace (and denote the corresponding projection by the same notation) for Λ with eigenvalue j. Then we have the orthogonal decomposition $H^2(S) = \oplus_{j\in\mathbf{N}}Q_j$. We call an operator T of **finite propagation** if there exists a positive number l such that

$$TQ_j \subset \bigoplus_{|i-j|\le l} Q_i.$$

Lemma 2.5. *Suppose the operator* T *has the finite propagation property. If* $T\Lambda^2$ *is bounded, then* $\Lambda^2 T$ *and* $T^*\Lambda^2$ *are also bounded.*

Proof. By assumption, we have that $T = \oplus_{-l\le i\le l}T_i$ for some number l, where

$$T_i = \oplus_j Q_{j+i}TQ_j$$

is an operator of degree i. By grading, one sees that each $\Lambda^2 T_i$ is bounded iff there exists a constant C_i such that $\|T_ip\| \le C_i\frac{\|p\|}{j^2}$ for any index j and $p\in Q_j$. Indeed, if such C_i exists, then for any $p = \oplus_j p_j$,

$$\|\Lambda^2 T_i p\|^2 = \sum_j \|(i+j)T_ip_j\|^2 \le C_i^2\sum_j \frac{(i+j)^2\|p_j\|^2}{j^2} \le C_i^2(1+l)^2\|p\|^2.$$

Using the fact that $T\Lambda^2$ is bounded, for each j and $p\in Q_j$, we have

$$\|T\Lambda^2 p\|^2 = j^2\|\sum_{-l\le i\le l} T_ip\|^2 = j^2\sum_{-l\le i\le l}\|T_ip\|^2 \ge j^2\|T_ip\|^2$$

for each i. It follows that each $\Lambda^2 T_i$ is bounded. Therefore $\Lambda^2 T = \sum\limits_{-l\le i\le l}\Lambda^2 T_i$ is bounded. This implies that $T^*\Lambda^2$ is bounded. ∎

Let $\mathcal{C}$ denote the $*$-algebra generated by T_p with polynomial symbol p, and $\frac{1}{\Lambda+t}$ together with all projections P^β, where $\beta\in\mathbf{N}_+^{r-1}$ is arbitrary. Define

$$\mathcal{B} := \{B\in\mathcal{C} : B\Lambda^2 \textit{ bounded}\},$$

$$\mathcal{B}_\Lambda := \mathcal{A}(\Lambda+1)^{-1} + \mathcal{B} = \{A(\Lambda+1)^{-1} + B :\ A\in\mathcal{A}, B\in\mathcal{B}\}.$$

It is easy to check that operators in $\mathcal{C}$ have the finite propagation property. Therefore Lemma (2.5) implies that $\mathcal{B}$ and $\mathcal{B}_\Lambda$ are invariant under taking adjoints.

Lemma 2.6. $\mathcal{B}$ *is an ideal in* $\mathcal{C}$*. Moreover,*

$$[\mathcal{C}, (\Lambda+t)^{-1}] \subset \mathcal{B}.$$

Proof. For the first assertion it suffices to show that $BT_u\in\mathcal{B}$ whenever $B\in\mathcal{B}$. Define a bounded operator R_u by $R_up = P_{m+1,\beta}T_up$ for $p\in\mathcal{P}_{m,\beta}(Z)$. Then

$$(\Lambda^2T_u - T_u\Lambda^2)p = \sum_{i=1}^r(\Lambda^2-m^2)P_{(m,\beta)+[i]}T_up = ((m+1)^2-m^2)P_{m+1,\beta}T_up = (2\Lambda-1)R_up.$$

Therefore $BT_u\Lambda^2 = B\Lambda^2T_u - B(2\Lambda-1)R_u$ is bounded. Thus $BT_u\in\mathcal{B}$. For the second assertion it suffices to show that $[T_u,(\Lambda+t)^{-1}]\in\mathcal{B}$. With the previous notation, we have

$$\begin{aligned}
[T_u,(\Lambda+t)^{-1}]\Lambda^2p &= m^2(T_u(\Lambda+t)^{-1} - (\Lambda+t)^{-1}T_u)p = m^2\sum_{i=1}^r\left(\frac{1}{m+t} - \frac{1}{\Lambda+t}\right)P_{(m,\beta)+[i]}T_up \\
&= m^2\left(\frac{1}{m+t} - \frac{1}{m+1+t}\right)P_{m+1,\beta}T_up = R_u\frac{\Lambda^2}{(\Lambda+t)(\Lambda+1+t)}p.
\end{aligned}$$

Therefore $[T_u,(\Lambda+t)^{-1}]\Lambda^2$ is bounded. ∎

Lemma 2.7. *$\mathcal{B}_\Lambda$ is a (non-unital) $*$-algebra and an $\mathcal{A}$-bimodule, i.e.,*

$$\mathcal{A}\mathcal{B}_\Lambda + \mathcal{B}_\Lambda\mathcal{A} \subset \mathcal{B}_\Lambda$$

.

Proof. We only show that $\mathcal{B}_\Lambda\mathcal{A} \subset \mathcal{B}_\Lambda$. Indeed, for $A \in \mathcal{A}, B \in \mathcal{B}$, and $u \in Z$, we have

$(A(\Lambda+1)^{-1} + B)S_u - AS_u(\Lambda+1)^{-1} = BS_u + A[(\Lambda+1)^{-1}, S_u] = BS_u - AS_u(\Lambda+1)^{-1}(\Lambda+2)^{-1} \in \mathcal{B}.$

Since $AS_u \in \mathcal{A}$, it follows that $A(\Lambda+1)^{-1} + B \in \mathcal{B}_\Lambda$. □

Proposition 2.8. *For $\lambda \in \mathbf{N}_+^r$ let $p, q \in \mathcal{P}(Z)$ satisfy $\deg(p), \deg(q) \leq |\lambda'|$. Then*

$$PT_p^* P^{\lambda'} T_q P \in \mathcal{A} + \mathcal{B}_\Lambda.$$

Proof. The K-invariant operator $P^{\lambda'} A^\lambda P^{\lambda'}$ is diagonal, and Proposition (1.6) implies that

$$P^{\lambda'} A^\lambda P^{\lambda'} = \frac{c_\lambda \Lambda + \tilde{c}_\lambda}{\Lambda + 1} P^{\lambda'} + B,$$

where $B \in \mathcal{B}$ and $c_\lambda > 0$. It follows that

$$P^{\lambda'} = P^{\lambda'} A^\lambda P^{\lambda'} \frac{\Lambda+1}{c_\lambda \Lambda + \tilde{c}_\lambda} + B'$$

with $B' \in \mathcal{B}$. If $\beta \in \mathbf{N}_+^{r-1}$ satisfies $|\beta| \leq |\lambda'|$ and $A^\lambda P^\beta$ is non-zero, then $\beta \geq \lambda'$ by Lemma (1.5). This is only possible if $\beta = \lambda'$. Therefore

$$PT_p^* A^\lambda T_q P = \sum_{|\beta| \leq |\lambda'|} PT_p^* P^\beta A^\lambda P^\beta T_q P = PT_p^* P^{\lambda'} A^\lambda P^{\lambda'} T_q P.$$

Since $\mathcal{B}$ is an ideal in $\mathcal{C}$ and $\left[\frac{\Lambda+1}{c_\lambda\Lambda+\tilde{c}_\lambda}, T_q P\right] \in \mathcal{B}$ we obtain

$$\begin{aligned} PT_p^* P^{\lambda'} T_q P &= PT_p^* P^{\lambda'} A^\lambda P^{\lambda'} \frac{\Lambda+1}{c_\lambda\Lambda+\tilde{c}_\lambda} T_q P + PT_p^* B' T_q P \\ &= PT_p^* P^{\lambda'} A^\lambda P^{\lambda'} T_q P \frac{\Lambda+1}{c_\lambda\Lambda+\tilde{c}_\lambda} + PT_p^* P^{\lambda'} A^\lambda P^{\lambda'} \left[\frac{\Lambda+1}{c_\lambda\Lambda+\tilde{c}_\lambda}, T_q P\right] + PT_p^* B' T_q P \\ &= PT_p^* A^\lambda T_q P \frac{\Lambda+1}{c_\lambda\Lambda+\tilde{c}_\lambda} + B'', \end{aligned}$$

where $B'' \in \mathcal{B}$. Since $PT_p^* A^\lambda T_q P \in \mathcal{A}$ by Corollary (2.3), the assertion follows. □

Proposition 2.9. $[\mathcal{A}, \mathcal{A}] \subset \mathcal{B}_\Lambda$.

Proof. In view of Lemma (2.7) it suffices to show that $[S_u^*, S_v] \in \mathcal{B}_\Lambda$. We may suppose that Z has rank $r > 1$. By definition, $S_v = PT_vP = T_vP - P^1T_vP$. Note $S_v\mathcal{P}_{m,0}(Z) \subset \mathcal{P}_{m+1,0}(Z)$ and $\frac{a}{2}(r-1)+b = \rho-1$. Applying (1.9) it follows that

$$\begin{aligned} (m+\rho)S_u^*S_vP_m = (m+\rho)T_u^*S_vP_m &= u^\partial(S_vP_m) = u^\partial(T_vP_m - P^1T_vP_m) \\ &= (u|v)P_m + T_vu^\partial P_m - u^\partial P^1T_vP_m \\ &= (u|v)P_m + (m+\rho-1)T_vS_u^*P_m - u^\partial P^1T_vP_m \\ &= (u|v)P_m + (m+\rho-1)S_vS_u^*P_m - Pu^\partial P^1T_vP_m. \end{aligned}$$

Thus $S_u^*S_v(\Lambda+\rho) = (u|v)P + S_vS_u^*(\Lambda+\rho-1) - Pu^\partial P^1T_vP$ and hence

$[S_u^*, S_v](\Lambda+\rho) = (u|v)P + S_vS_u^*((\Lambda+\rho-1) - (\Lambda+\rho)) - Pu^\partial P^1T_vP = (u|v)P - S_vS_u^* - Pu^\partial P^1T_vP.$

By (1.9) we have

$$\begin{aligned} Pu^\partial P^1T_vP = \sum_m Pu^\partial P^1T_vP_m = \sum_m P_mu^\partial P_{m,1}T_vP_m = (1+\frac{a}{2}(r-2)+b)\sum_m P_mT_u^*P_{m,1}T_vP_m \\ = (1+\frac{a}{2}(r-2)+b)\sum_m PT_u^*P^1T_vP_m = (1+\frac{a}{2}(r-2)+b)PT_u^*P^1T_vP. \end{aligned}$$

Thus Proposition (2.8) implies that $Pu^\partial P^1 T_v P \in \mathcal{A}+\mathcal{B}_\Lambda$, and the assertion follows. $\square$

Lemma 2.10. $\mathcal{A} \subset \left\{ \sum_i S_{p_i} S_{q_i}^* + B : \, p_i, q_i \in \mathcal{P}(Z), \, B \in \mathcal{B}_\Lambda \right\}$.

Proof. Since the latter set contains S_u, S_v^*, it suffices to show that it is invariant under multiplication by S_u, S_v^*. By Proposition (2.9) we have $[S_u^*, S_p] \in \mathcal{B}_\Lambda$ and $[S_q^*, S_v] \in \mathcal{B}_\Lambda$. With Lemma (2.7), the assertion follows. $\square$

The following technical lemma will be used in the next section.

Lemma 2.11. *Let $T \in \mathcal{A}+\mathcal{B}_\Lambda$. Then* $\left\{ \frac{(N_1^m|TN_1^m)_S}{\|N_1^m\|_S^2} \right\}_m \in \mathcal{S}$.

Proof. By Lemma (1.4), we have

$$S_{N_1^k}^* N_1^m = \frac{(m+1)_k^*}{(m+\rho)_k^*} N_1^{m-k}$$

for $0 \le k \le m$, $S_{N_1^k}^* N_1^m = 0$ for $k > m$ and $S_v^* N_1^m = 0$ for all $v \in Z_1^\perp$. Thus for any $p, q \in \mathcal{P}(Z)$ there exist constants $c_k(p,q)$, for $0 \le k \le M(p,q) := min(\deg p, \deg q)$, such that

$$(S_p^* N_1^m | S_q^* N_1^m)_S = \sum_{k=0}^{M(p,q)} c_k(p,q) \|S_{N_1^k}^* N_1^m\|_S^2 = \sum_{k=0}^{M(p,q)} c_k(p,q) \|\frac{(m+1)_k^*}{(m+\rho)_k^*} N_1^{m-k}\|_S^2$$

for all $m \ge M(p,q)$. Since $T \in \mathcal{A}+\mathcal{B}_\Lambda$, Lemma (2.10) implies that

$$T = \sum_i S_{p_i} S_{q_i}^* + B_0$$

for some polynomials p_i, q_i and $B_0 = A_1(\Lambda+1)^{-1} + B_1 \in \mathcal{B}_\Lambda$ with $A_1 \in \mathcal{A}, B_1 \in \mathcal{B}$. Using Lemma (2.10) for A_1 again, there exist polynomials ϕ_j, ψ_j and $B_2 \in \mathcal{B}_\Lambda$ such that

$$T = \sum_i S_{p_i} S_{q_i}^* + \left(\sum_j S_{\phi_j} S_{\psi_j}^* + B_2 \right) (\Lambda+1)^{-1} + B_1 = \sum_i S_{p_i} S_{q_i}^* + \sum_j S_{\phi_j} S_{\psi_j}^* (\Lambda+1)^{-1} + B,$$

where $B \in \mathcal{B}$. It follows that

$$(N_1^m|TN_1^m)_S - (N_1^m|BN_1^m)_S = \sum_i (S_{p_i}^* N_1^m | S_{q_i}^* N_1^m)_S + \frac{1}{m+1} \sum_j (S_{\phi_j}^* N_1^m | S_{\psi_j}^* N_1^m)_S$$

$$= \sum_i \sum_{k=0}^{M(p_i,q_i)} c_k(p_i,q_i) \, \|\frac{(m+1)_k^*}{(m+\rho)_k^*} N_1^{m-k}\|_S^2 + \frac{1}{m+1} \sum_j \sum_{k=0}^{M(\phi_j,\psi_j)} c_k(\phi_j,\psi_j) \, \|\frac{(m+1)_k^*}{(m+\rho)_k^*} N_1^{m-k}\|_S^2.$$

Since

$$BN_1^m = (B(\Lambda+1)^2)(\Lambda+1)^{-2} N_1^m = \frac{B(\Lambda+1)^2 N_1^m}{(m+1)^2}$$

the sequence $\{m^2 \frac{(N_1^m|BN_1^m)_S}{\|N_1^m\|_S^2}\}$ is bounded. Thus there exist finitely many sequences $\{c_k(m)\} \in \mathcal{S}$ such that

$$\frac{(N_1^m|TN_1^m)_S}{\|N_1^m\|_S^2} = \sum_k c_k(m) \frac{\|N_1^{m-k}\|_S^2}{\|N_1^m\|_S^2}.$$

This yields the desired result since $\left\{ \frac{\|N_1^{m-k}\|_S^2}{\|N_1^m\|_S^2} \right\}_m \in \mathcal{S}_+$ by Lemma (1.3). $\square$

3. First main theorem

Theorem 3.1. *Let $f \in \mathcal{P}(Z \times \overline{Z})$ be a real-analytic polynomial. Then $S_f \in \mathcal{A} + \mathcal{B}_\Lambda$.*

The proof is based on a lengthy induction argument. We may assume that $f = \overline{p}\,q$ for some $p, q \in \mathcal{P}(Z)$. Let $\mathcal{A}_{i,j}$ denote the set of all operators $PT_p^* T_q P$, where $\deg p \le i$, $\deg q \le j$. For a given k we consider the following assumption

$$\mathcal{A}_{i,j} \subset \mathcal{A} + \mathcal{B}_\Lambda \text{ whenever } min(i,j) < k. \tag{3.1}$$

We now proceed via a sequence of 'claims' which are proved under this assumption.

Claim 3.2. *The assumption* (3.1) *implies that for each partition λ with $|\lambda| < k$ there exist constants $a_\beta^\lambda, b_\beta^\lambda$ such that*

$$A^\lambda - \sum_{\beta \le \lambda} \frac{a_\beta^\lambda \Lambda + b_\beta^\lambda}{\Lambda + 1} P^\beta \in \mathcal{B}. \tag{3.2}$$

Proof. For $\beta \le \lambda \le (m, \beta)$ we have $N_{m,\beta} = N_{\beta_1,\beta} N_1^{m-\beta_1}$ and hence

$$\frac{(N_{m,\beta}|A^\lambda N_{m,\beta})_S}{\|N_{m,\beta}\|_S^2} = \sum_i \frac{\|PT_{p_i}^* N_{m,\beta}\|_S^2}{\|N_{m,\beta}\|_S^2} = \sum_i \frac{\|PT_{p_i}^* T_{N_{\beta_1,\beta}} N_1^{m-\beta_1}\|_S^2}{\|N_1^{m-\beta_1}\|_S^2} \frac{\|N_1^{m-\beta_1}\|_S^2}{\|N_{m,\beta}\|_S^2}.$$

Since $\deg(p_i) = |\lambda| < k$, (3.1) implies $PT_{p_i}^* T_{N_{\beta_1,\beta}} P \in \mathcal{A} + \mathcal{B}_\Lambda$. By Lemma (2.11) and Lemma (1.3), we have that $\left\{ \frac{(N_{m,\beta}|A^\lambda N_{m,\beta})_S}{\|N_{m,\beta}\|_S^2} \right\}_m \in \mathcal{S}$. By (1.12) there is a sequence $\mathfrak{o}_m$, with $m^2\, \mathfrak{o}_m$ bounded, such that

$$A^\lambda = \sum_{\beta \le \lambda \le (m,\beta)} \frac{(N_{m,\beta}|A^\lambda N_{m,\beta})_S}{\|N_{m,\beta}\|_S^2} P_{m,\beta} = \sum_{\beta \le \lambda \le (m,\beta)} \left(\frac{a_\beta^\lambda m + b_\beta^\lambda}{m+1} + \mathfrak{o}_m \right) P_{m,\beta} = \sum_{\beta \le \lambda} \frac{a_\beta^\lambda \Lambda + b_\beta^\lambda}{\Lambda + 1} P^\beta + B,$$

where we set $a_\beta^\lambda = b_\beta^\lambda = 0$ if $\beta \not\ge \lambda'$. Thus $B - \sum\limits_{\beta \le \lambda \le (m,\beta)} \mathfrak{o}_m\, P_{m,\beta}$ has finite rank and hence $B \in \mathcal{B}$. ☐

Claim 3.3. *Under the assumption* (3.1) *there exist constants $c_\alpha^\beta, d_\alpha^\beta$ such that*

$$P^\beta - \sum_{\alpha \le \beta} \frac{c_\alpha^\beta \Lambda + d_\alpha^\beta}{\Lambda + 1} A^\alpha \in \mathcal{B}, \qquad \forall\, \beta \in \mathbf{N}_+^{r-1},\ |\beta| < k. \tag{3.3}$$

Proof. We use induction on $|\beta|$. The case $\beta = 0$ is trivial. Assume (3.3) holds for all β with $|\beta| < j < k$. Let β satisfy $|\beta| = j$. Then Claim (3.2) implies

$$A^\beta = \frac{a_\beta^\beta \Lambda + b_\beta^\beta}{\Lambda + 1} P^\beta + \sum_{\alpha < \beta} \frac{a_\alpha^\beta \Lambda + b_\alpha^\beta}{\Lambda + 1} P^\alpha + B^\beta, \tag{3.4}$$

where $B^\beta \in \mathcal{B}$, and $\alpha < \beta$ means that $\alpha \le \beta$ and $\alpha \ne \beta$. Now consider the diagonal operator

$$\sum_{|\lambda| = j} A^\lambda = \sum_{\mu \in \mathbf{N}_+^r} a_\mu P_\mu.$$

If $|\lambda| = j$, then $(N_{m,\beta}|A^\lambda N_{m,\beta})_S$ is non-zero only if $\lambda = \beta$, since $\beta \le \lambda$ and $|\beta| = j = |\lambda|$. Therefore

$$a_{m,\beta}\, N_{m,\beta} = \sum_{|\lambda| = j} A^\lambda N_{m,\beta} = \sum_{|\lambda| = j} \frac{(N_{m,\beta}|A^\lambda N_{m,\beta})_S}{\|N_{m,\beta}\|_S^2}\, N_{m,\beta} = \frac{(N_{m,\beta}|A^\beta N_{m,\beta})_S}{\|N_{m,\beta}\|_S^2}\, N_{m,\beta}.$$

By [24, Theorem 1.6], there exists a constant $c > 0$ such that $a_\mu \ge c$ whenever $\mu_2 + \cdots + \mu_r = j$. For $\mu = (m, \beta)$ this implies $\frac{(N_{m,\beta}|A^\beta N_{m,\beta})_S}{\|N_{m,\beta}\|_S^2} = a_{m,\beta} \ge c$ and hence $a_\beta^\beta = \lim_m \frac{(N_{m,\beta}|A^\beta N_{m,\beta})_S}{\|N_{m,\beta}\|_S^2} \ge c > 0$. For any $|\alpha| < |\beta| = j$, the induction hypothesis implies

$$P^\alpha = \sum_{\gamma \le \alpha} \frac{c_\gamma^\alpha \Lambda + d_\gamma^\alpha}{\Lambda + 1} A^\gamma + B^\alpha,$$

where $B^\alpha \in \mathcal{B}$. Plugging into (3.4) we obtain

$$P^\beta = \frac{\Lambda+1}{a^\beta_\beta\Lambda+b^\beta_\beta}\left[A^\beta - \sum_{\alpha<\beta}\frac{a^\beta_\alpha\Lambda+b^\beta_\alpha}{a^\beta_\beta\Lambda+b^\beta_\beta}\left(\sum_{\gamma\le\alpha}\frac{c^\alpha_\gamma\Lambda+d^\alpha_\gamma}{\Lambda+1}A^\gamma + B^\alpha\right) - B^\beta\right].$$

It is easy to see that this expression has the desired form. □

Claim 3.4. *The assumption* (3.1) *implies* $\mathcal{A}_{k,k} \subset \mathcal{A} + \mathcal{B}_\Lambda$.

Proof. Let $\deg p = \deg q = k$. Then

$$PT_p^* T_q P = \sum_{|\beta|\le k} PT_p^* P^\beta T_q P.$$

If $|\beta| = k$ then $PT_p^* P^\beta T_q P \in \mathcal{A} + \mathcal{B}_\Lambda$ by Proposition (2.8). If $|\beta| = h < k$ and $\alpha \le \beta$ then (3.1) implies $PT_p^* A^\alpha T_q P \in \mathcal{A}_{k,h}\mathcal{A}_{h,k} \subset \mathcal{A} + \mathcal{B}_\Lambda$. It follows that

$$PT_p^* A^\alpha \frac{c^\beta_\alpha\Lambda+d^\beta_\alpha}{\Lambda+1} T_q P = PT_p^* A^\alpha T_q P \frac{c^\beta_\alpha\Lambda+d^\beta_\alpha}{\Lambda+1} + PT_p^* A^\alpha\left[\frac{c^\beta_\alpha\Lambda+d^\beta_\alpha}{\Lambda+1}, T_q P\right] \in \mathcal{A} + \mathcal{B}_\Lambda,$$

since $\left[\frac{c^\beta_\alpha\Lambda+d^\beta_\alpha}{\Lambda+1}, \mathcal{C}\right] \subset \mathcal{B}$ and $\mathcal{B}$ is an ideal in $\mathcal{C}$. Therefore Claim (3.3) implies $PT_p^* P^\beta T_q P \in \mathcal{A} + \mathcal{B}_\Lambda$. □

Claim 3.5. *Under the assumption* (3.1)*, for* $T \in \mathcal{C}$ *and* $q \in \mathcal{P}(Z)$ *of degree* $i < k$ *there exists* $B \in \mathcal{B}$ *such that*

$$PT[T_u^*, T_v]T_q P = B + \sum_{|\beta|\le i}\sum_{\alpha\le\beta}\sum_{\gamma\le\beta} PTA^\alpha[T_u^*, T_v]A^\gamma T_q P \frac{c^\beta_\alpha c^\beta_\gamma\Lambda + c^\beta_\alpha(d^\beta_\gamma - c^\beta_\gamma) + (d^\beta_\alpha - c^\beta_\alpha)c^\beta_\gamma}{\Lambda+1}.$$

Proof. Since $\mathrm{Ran}(T_q P) \subset \sum_{|\beta|\le i} P^\beta$ and $[T_u^*, T_v]$ is a 'block-diagonal' operator [24, Lemma 2.1] which commutes with each P^β, it suffices to consider $PTP^\beta[T_u^*, T_v]P^\beta T_q P$ for $\beta \in \mathbf{N}_+^{r-1}$ satisfying $|\beta| \le i$. By Claim (3.3) we have

$$PTP^\beta[T_u^*, T_v]P^\beta T_q P = PT\left(B_1 + \sum_{\alpha\le\beta}\frac{c^\beta_\alpha\Lambda+d^\beta_\alpha}{\Lambda+1}A^\alpha\right)[T_u^*, T_v]\left(B_2 + \sum_{\gamma\le\beta}\frac{c^\beta_\gamma\Lambda+d^\beta_\gamma}{\Lambda+1}A^\gamma\right)T_q P,$$

where $B_1, B_2 \in \mathcal{B}$. Since $\mathcal{B} \subset \mathcal{C}$ is an ideal and $\mathcal{C}$ contains PT, $[T_u^*, T_v]$, A^α, A^γ, $T_q P$, the assertion follows. □

Claim 3.6. *The assumption* (3.1) *implies*

$$PT_\phi^*[T_u^*, T_v]T_\psi P \in \mathcal{A} + \mathcal{B}_\Lambda \tag{3.5}$$

whenever $\deg\phi, \deg\psi < k$.

Proof. We prove (3.5) by induction on $h = \max(\deg\phi, \deg\psi) < k$. For $h = 0$, we have

$$P[T_u^*, T_v]P = PT_u^* T_v P - PT_v PT_u^* P,$$

where $PT_u^* T_v P \in \mathcal{A}_{1,1} \subset \mathcal{A} + \mathcal{B}_\Lambda$ by Claim (3.4), and $PT_v PT_u^* P \in \mathcal{A}$. For the induction step, let ϕ, ψ be polynomials with $\deg\phi \le h = \deg\psi < k$, and we may assume that (3.5) holds in the case of the maximal degree less than h. Then

$$PT_\phi^*[T_u^*, T_v]T_\psi P = PT_\phi^*\left([T_u^*, T_{v\psi}] - T_v[T_u^*, T_\psi]\right)P = PT_{\phi u}^* T_{v\psi} P - PT_\phi^* T_{v\psi} PT_u^* P - PT_\phi^* T_v[T_u^*, T_\psi]P.$$

By the assumption (3.1), we have $PT_\phi^* T_{v\psi} P \in \mathcal{A}_{h,h+1} \subset \mathcal{A}+\mathcal{B}_\Lambda$, and using Claim (3.4) also $PT_{\phi u}^* T_{v\psi} P \in \mathcal{A}_{h+1,h+1} \subset \mathcal{A} + \mathcal{B}_\Lambda$. For the third term we may assume that $\psi = v_{h-1}\cdots v_0$ for some linear functions v_i. Then

$$PT_\phi^* T_v[T_u^*, T_\psi]P = \sum_{i=0}^{h-1} PT_\phi^* T_{v\cdot v_{h-1}\cdots v_{i+1}}[T_u^*, T_{v_i}]T_{v_{i-1}\cdots v_0}P.$$

If p, q, ξ, η are polynomials of degree $\leq i < h$ we have

$$PT_\phi^* T_{v \cdot v_{h-1} \cdots v_{i+1}} T_p P T_q^* [T_u^*, T_{v_i}] T_\xi P T_\eta^* T_{v_{i-1} \cdots v_0} P \in \mathcal{A} + \mathcal{B}_\Lambda,$$

since (3.1) implies that $\mathcal{A} + \mathcal{B}_\Lambda$ contains $PT_\phi^* T_{v \cdot v_{h-1} \cdots v_{i+1}} T_p P \in \mathcal{A}_{h,h}$ and $PT_\eta^* T_{v_{i-1} \cdots v_0} P \in \mathcal{A}_{i,i}$, and the induction hypothesis implies $PT_q^* [T_u^*, T_{v_i}] T_\xi P \in \mathcal{A} + \mathcal{B}_\Lambda$. Thus

$$PT_\phi^* T_{v \cdot v_{h-1} \cdots v_{i+1}} A^\alpha [T_u^*, T_{v_i}] A^\rho T_{v_{i-1} \cdots v_0} P \subset \mathcal{A} + \mathcal{B}_\Lambda,$$

whenever $|\alpha| \leq i$ and $|\rho| \leq i$. Now the assertion follows from Claim (3.5) □

The **proof of Theorem** (3.1) can now be completed as follows. Since $\mathcal{A}_{m,n}^* = \mathcal{A}_{n,m}$, it suffices to show that

$$\mathcal{A}_k := \sum_{\ell \geq k} \mathcal{A}_{\ell,k} \subset \mathcal{A} + \mathcal{B}_\Lambda. \tag{3.6}$$

We prove (3.6) by induction over $k \geq 0$. The case $k = 0$ is trivial. For the induction step, let $k > 0$ and suppose that $\mathcal{A}_h \subset \mathcal{A} + \mathcal{B}_\Lambda$ whenever $h < k$. This is precisely the assumption (3.1). We prove that

$$\mathcal{A}_{\ell,k} \subset \mathcal{A} + \mathcal{B}_\Lambda \tag{3.7}$$

by induction over $\ell \geq k$. By Claim (3.4) we have $\mathcal{A}_{k,k} \subset \mathcal{A} + \mathcal{B}_\Lambda$. For the induction step assume that $\mathcal{A}_{\ell,k} \subset \mathcal{A} + \mathcal{B}_\Lambda$ for some $\ell \geq k$. Passing to $\ell + 1$, consider polynomials ϕ, ψ with $\deg \phi \leq \ell$ and $\deg \psi = k$. Then we have for any linear function u

$$PT_{\phi \cdot u}^* T_\psi P = PT_\phi^* T_u^* T_\psi P = PT_\phi^* T_\psi P T_u^* P + PT_\phi^* [T_u^*, T_\psi] P.$$

By the induction hypothesis we have $PT_\phi^* T_\psi P \in \mathcal{A}_{\ell,k} \subset \mathcal{A} + \mathcal{B}_\Lambda$. For the second term, we may assume that $\psi = v_{k-1} \cdots v_0$ for some linear functions v_i. Then

$$PT_\phi^* [T_u^*, T_\psi] P = \sum_{i=0}^{k-1} PT_\phi^* T_{v_{k-1} \cdots v_{i+1}} [T_u^*, T_{v_i}] T_{v_{i-1} \cdots v_0} P.$$

If p, q, ξ, η are polynomials of degree $\leq i < k$ we have

$$PT_\phi^* T_{v_{k-1} \cdots v_{i+1}} T_p P T_q^* [T_u^*, T_{v_i}] T_\xi P T_\eta^* T_{v_{i-1} \cdots v_0} P \in \mathcal{A} + \mathcal{B}_\Lambda,$$

since the assumption (3.1) implies that $\mathcal{A} + \mathcal{B}_\Lambda$ contains $PT_\phi^* T_{v_{k-1} \cdots v_{i+1}} T_p P \in \mathcal{A}_{\ell,k-1}$ and $PT_\eta^* T_{v_{i-1} \cdots v_0} P \in \mathcal{A}_{i,i}$, and Claim (3.6) implies $PT_q^* [T_u^*, T_{v_i}] T_\xi P \in \mathcal{A} + \mathcal{B}_\Lambda$. Thus

$$PT_\phi^* T_{v_{k-1} \cdots v_{i+1}} A^\alpha [T_u^*, T_{v_i}] A^\gamma T_{v_{i-1} \cdots v_0} P \in \mathcal{A} + \mathcal{B}_\Lambda,$$

whenever $|\alpha| \leq i$ and $|\gamma| \leq i$. With Claim (3.5), it follows that $PT_\phi^* [T_u^*, T_\psi] P \in \mathcal{A} + \mathcal{B}_\Lambda$. Therefore $\mathcal{A}_{\ell+1,k} \subset \mathcal{A} + \mathcal{B}_\Lambda$, completing the induction proof of (3.6).

4. Smooth extension and Dixmier trace

Let $\mathcal{K}$ denote the compact operators. By definition [6] we have

$$\mathcal{L}^{n,\infty} := \{T \in \mathcal{K} : \mu_j(T) = O(j^{-1/n})\}$$

for $n > 1$, and

$$\mathcal{L}^{1,\infty} := \{T \in \mathcal{K} : \sum_{i=1}^{j} \mu_i(T) = O(\log j)\}.$$

Here $\mu_1(T) \geq \mu_2(T) \geq \cdots$ are the singular values of T. We will apply these concepts to the Hilbert space $H_1^2(S)$. Using the invariants a, b we put

$$n := 1 + a(r-1) + b.$$

Note that n is not the dimension $d = r(1 + \frac{a}{2}(r-1) + b)$ of the underlying domain D, unless $r = 1$. We will give a geometric interpretation below.

Lemma 4.1. *Consider Λ as an unbounded operator on $H_1^2(S)$. Then $(\Lambda + 1)^{-1} \in \mathcal{L}^{n,\infty}$.*

Proof. For any partition λ it follows from [23, Lemma 2.7 and Lemma 2.6] that

$$\dim \mathcal{P}_\lambda(Z) = \frac{(\rho)_\lambda}{(\rho-b)_\lambda} \prod_{1\le i<j\le r} \frac{\lambda_i-\lambda_j+\frac{a}{2}(j-i)}{\frac{a}{2}(j-i)} \cdot \frac{(\lambda_i-\lambda_j+1+\frac{a}{2}(j-i-1))_{a-1}}{(1+\frac{a}{2}(j-i-1))_{a-1}}. \tag{4.1}$$

Specializing (4.1) to $m=(m,0,\dots,0)$ we obtain

$$\dim \mathcal{P}_m(Z) = \frac{(m+1+\frac{a}{2}(r-1))_b}{(1+\frac{a}{2}(r-1))_b} \prod_{j=2}^{r} \frac{m+\frac{a}{2}(j-1)}{\frac{a}{2}(j-1)} \cdot \frac{(m+1+\frac{a}{2}(j-2))_{a-1}}{(1+\frac{a}{2}(j-2))_{a-1}}$$

for $m \ge b$. It follows that asymptotically, we have

$$\dim \mathcal{P}_m(Z) \sim c\cdot m^{b+a(r-1)} = c\cdot m^{n-1}$$

for some constant $c>0$ independent of m. Since $(\Lambda+1)^{-1}$ has the eigenvalues $1/(1+m)$, with eigenspace $\mathcal{P}_m(Z)$, this estimate implies that the partial sum

$$S_j((\Lambda+1)^{-1}) = \sum_{i=0}^{j} \mu_i((\Lambda+1)^{-1}) \sim j^{1-1/n},$$

where $\mu_i(T)$ is the i-th eigenvalue of T. This implies the assertion since, for $n>1$, $T\in\mathcal{L}^{n,\infty}$ iff $\{j^{(1/n-1)}S_j(T) : j\ge 1\}$ is a bounded sequence [6]. □

Theorem 4.2. *Let $f,g\in\mathcal{P}(Z\times\overline{Z})$ be real-analytic polynomials. Then $[S_f,S_g]\in\mathcal{L}^{n,\infty}$.*

Proof. Let $A\in\mathcal{A}, B\in\mathcal{B}$. Then Lemma (4.1) implies $A(\Lambda+1)^{-1}+B\in\mathcal{L}^{n,\infty}$ since $A+B(\Lambda+1)$ is bounded. It follows that $\mathcal{B}_\Lambda\subset\mathcal{L}^{n,\infty}$. Since $[\mathcal{A},\mathcal{A}]\subset\mathcal{B}_\Lambda$ by Proposition (2.9) and $\mathcal{B}_\Lambda$ is an $\mathcal{A}$-bimodule we obtain

$$[\mathcal{A}+\mathcal{B}_\Lambda, \mathcal{A}+\mathcal{B}_\Lambda]\subset\mathcal{B}_\Lambda.$$

Since $S_f,S_g\in\mathcal{A}+\mathcal{B}_\Lambda$ by Theorem (3.1), the assertion follows. □

It is well known [6] that $T_i\in\mathcal{L}^{p_i,\infty}$ and $\sum\limits_{i=1}^{n}\frac{1}{p_i}=1$ implies $T=T_1\cdots T_n\in\mathcal{L}^{1,\infty}$. Hence Theorem (4.2) implies

Corollary 4.3. *Let $f_1,g_1,\dots f_n,g_n\in\mathcal{P}(Z\times\overline{Z})$ be real-analytic polynomials. Then*

$$[S_{f_1},S_{g_1}]\cdots[S_{f_n},S_{g_n}]\in\mathcal{L}^{1,\infty}. \tag{4.2}$$

The trace class $\mathcal{L}^1$ is a proper subspace of $\mathcal{L}^{1,\infty}$. For $T\in\mathcal{L}^{1,\infty}$ the **Dixmier trace**, denoted by $tr_\omega(T)$, depends a priori on a choice of positive functional ω on $l^\infty(\mathbf{N})$ vanishing on $c_0(\mathbf{N})$. For the so-called **measurable** operators T the value $tr_\omega(T)$ is independent of ω. More precisely, for a positive operator T,

$$tr_\omega(T) = \lim_{j\to\infty}\frac{1}{\log j}\sum_{i=1}^{j}\mu_i(T)$$

whenever the limit exists. It also satisfies the tracial property

$$tr_\omega(TS) = tr_\omega(ST)$$

and $tr_\omega(T)=0$ if $T\in\mathcal{L}^1$. We refer the reader to [6] for more details.

In order to determine the Dixmier trace of the operators (4.2) we consider the algebraic variety

$$Z_1^\bullet := \{z\in Z : rank(z)\le 1\},$$

which has (complex) dimension $\dim Z_1^\bullet = 1+a(r-1)+b=n$, and is singular only at the origin.

Proposition 4.4. *Consider the polynomial ideal $\mathcal{I}(Z_1^\bullet)\subset\mathcal{P}(Z)$ vanishing on $Z_1^\bullet$. Then the sub-Hardy space $H_1^2(S)$ can be identified with the Hilbert quotient module*

$$H_1^2(S) = H^2(S)/\overline{\mathcal{I}(Z_1^\bullet)} \approx \overline{\mathcal{I}(Z_1^\bullet)}^{\perp}.$$

Proof. It suffices to show that $\mathcal{I}(Z_1^\bullet)$ coincides with the ideal

$$\mathcal{J} = \bigoplus_{\lambda_2>0} \mathcal{P}_\lambda(Z) \subset \mathcal{P}(Z).$$

For $z \in Z_1^\bullet$, we have $N_\ell(z) = 0$ for $\ell \geq 2$ since $rank(z) \leq 1$. This implies that $\mathcal{J} \subset \mathcal{I}(Z_1^\bullet)$. By Schur orthogonality the orthogonal projection P_λ is given by

$$\frac{1}{d_\lambda} P_\lambda\, f = \sum_\alpha \int_K dk\, (\phi_\alpha | k \cdot \phi_\alpha)\, (k^{-1} \cdot f)$$

for all $f \in \mathcal{P}(Z)$, where $\phi_\alpha \in \mathcal{P}_\lambda(Z)$ is an orthonormal basis. It follows that the K-invariant ideal $\mathcal{I}(Z_1^\bullet)$ is invariant under all P_λ. Now suppose there exists $f \in \mathcal{I}(Z_1^\bullet) \setminus \mathcal{J}$. Then $f = f' + f''$, where $f'' \in \mathcal{J}$ and $f' \in \mathcal{J}^\perp = \bigoplus_m \mathcal{P}_m(Z)$ is non-zero. Since $\mathcal{J} \subset \mathcal{I}(Z_1^\bullet)$ we may assume $f = f'$. By the above, we may assume that $f \in \mathcal{P}_m(Z)$ for some $m \geq 0$. By irreducibiliy, it follows that $\mathcal{P}_m(Z) \subset \mathcal{I}(Z_1^\bullet)$, which is a contradiction since $N_1^m \notin \mathcal{I}(Z_1^\bullet)$. □

The unit ball $D \cap Z_1^\bullet$ of $Z_1^\bullet$ is a **strictly pseudo-convex domain** (singular at the origin), with a K-homogeneous smooth boundary $S_1 = \{c : \{cc^*c\} = c, rank(c) = 1\}$ consisting of all minimal tripotents. Denote by $L^2(S_1)$ the L^2-space with respect to the K-invariant measure. The Hardy space $H^2(S_1)$ is the closure of the algebra $\mathcal{P}(Z)$ of all polynomials on Z, restricted to S_1. Since $N_\ell|_{S_1} = 0$ for each $\ell \geq 2$, it follows that

$$H^2(S_1) = \sum_{m\geq 0} \mathcal{P}_m^\sim(Z),$$

where $\tilde{f} = f|_{S_1}$ denotes the restriction.

Lemma 4.5. *Let $p, q \in \mathcal{P}_m(Z)$. Then*

$$(p|q)_S = \frac{(ra/2)_m}{(a/2)_m} (\tilde{p}|\tilde{q})_{S_1}.$$

Hence the transformation $U : H_1^2(S) \to H^2(S_1)$, defined by

$$Up := \sqrt{\frac{(ra/2)_m}{(a/2)_m}}\,\tilde{p} \qquad \forall\, p \in \mathcal{P}_m(Z),$$

is unitary.

Proof. Let X be the self-adjoint part of the Peirce 2-space Z_e^2 of full rank r [14]. For any partition $\lambda \in \mathbf{N}_+^r$, the associated **spherical polynomial** ϕ^λ on $X \subset Z$, normalized by $\phi^\lambda(e) = 1$ [14], is given by

$$\frac{\mathcal{E}^\lambda(t, e)}{d_\lambda} = \frac{\phi^\lambda(t)}{(d/r)_\lambda}$$

for all $t \in X$, where $d_\lambda := \dim \mathcal{P}_\lambda(Z)$ and $\mathcal{E}^\lambda(z, w)$ is the Fischer-Fock reproducing kernel for $\mathcal{P}_\lambda(Z)$. Now suppose $\lambda \in \mathbf{N}_+^\ell$. Then [2, Proposition 3.7] implies

$$\phi^\lambda(e_1 + \ldots + e_\ell) = \frac{(\ell a/2)_\lambda}{(ra/2)_\lambda}$$

and hence

$$\phi^\lambda(t) = \frac{(\ell a/2)_\lambda}{(ra/2)_\lambda} \phi_\ell^\lambda(t)$$

for all $t \in X_\ell \subset X$, where ϕ_ℓ^λ is the spherical polynomial for the self-adjoint part X_ℓ of the Peirce 2-space $Z^2_{e_1+\ldots e_\ell}$. Let $\Omega_\ell \subset X_\ell$ be the strictly positive cone, and let $t \in \Omega_\ell$ be fixed. By Schur orthogonality [7, Theorem 14.3.3], we have

$$\int_K dk\, \mathcal{E}^\lambda(z, k\sqrt{t}) \mathcal{E}^\mu(k\sqrt{t}, w) = \frac{\delta_{\lambda,\mu}}{d_\lambda} \mathcal{E}^\lambda(k\sqrt{t}, k\sqrt{t})\, \mathcal{E}^\lambda(z, w)$$

$$= \frac{\delta_{\lambda,\mu}}{(d/r)_\lambda}\phi^\lambda(t)\,\mathcal{E}^\lambda(z,w) = \frac{\delta_{\lambda,\mu}}{(d/r)_\lambda}\frac{(\ell a/2)_\lambda}{(ra/2)_\lambda}\phi^\lambda_\ell(t)\mathcal{E}^\lambda(z,w).$$

for all $\lambda,\mu \in \mathbf{N}^\ell_+$ and $z,w \in Z$. Applying this identity to $\ell = r, t = e$ and $\ell = 1, t = e_1$, resp., the assertion follows. $\square$

Define $\tilde{\Lambda}\tilde{p} = m\tilde{p}$ for $p \in \mathcal{P}_m(Z)$. Then Lemma (4.1) gives

$$\frac{1}{1+\tilde{\Lambda}} \in \mathcal{L}^{n,\infty}.$$

Let $\tilde{T}_f$ denote the Toeplitz operators $H^2(S_1)$. Then $\tilde{T}_u\tilde{\Lambda} = (\tilde{\Lambda}-1)\tilde{T}_u$ and $\tilde{T}^*_u\tilde{\Lambda} = (\tilde{\Lambda}+1)\tilde{T}^*_u$.

Proposition 4.6. *Let $u,v \in Z$. Then $US_uU^* - \tilde{T}_u \in \mathcal{L}^{n,\infty}$ and*

$$U[S_u,S_v^*]U^* - [\tilde{T}_u,\tilde{T}^*_v] \in \mathcal{L}^{n/2,\infty}.$$

Proof. For each $p \in \mathcal{P}_m(Z)$ we have

$$\begin{aligned} US_uU^*\tilde{p} &= U^*PT_u\left(\sqrt{\frac{(a/2)_m}{(ra/2)_m}}p\right) = \sqrt{\frac{(a/2)_m}{(ra/2)_m}}U^*P(up) \\ &= \sqrt{\frac{(a/2)_m}{(ra/2)_m}}\sqrt{\frac{(ra/2)_{m+1}}{(a/2)_{m+1}}}\widetilde{P(up)} = \sqrt{\frac{ra/2+m}{a/2+m}}\tilde{u}\tilde{p} = \tilde{T}_u\sqrt{\frac{\tilde{\Lambda}+ra/2}{\tilde{\Lambda}+a/2}}\tilde{p}. \end{aligned}$$

Thus we have $US_uU^* = \tilde{T}_u\sqrt{\frac{\tilde{\Lambda}+ra/2}{\tilde{\Lambda}+a/2}}$. This implies the first assertion. For the second assertion

$$\begin{aligned} U[S_u,S_v^*]U^* &= \left[\tilde{T}_u\sqrt{\frac{\tilde{\Lambda}+ra/2}{\tilde{\Lambda}+a/2}}, \sqrt{\frac{\tilde{\Lambda}+ra/2}{\tilde{\Lambda}+a/2}}\tilde{T}^*_v\right] = \tilde{T}_u\frac{\tilde{\Lambda}+ra/2}{\tilde{\Lambda}+a/2}\tilde{T}^*_v - \sqrt{\frac{\tilde{\Lambda}+ra/2}{\tilde{\Lambda}+a/2}}\tilde{T}^*_v\tilde{T}_u\sqrt{\frac{\tilde{\Lambda}+ra/2}{\tilde{\Lambda}+a/2}} \\ &= \frac{\tilde{\Lambda}+ra/2-1}{\tilde{\Lambda}+a/2-1}\tilde{T}_u\tilde{T}^*_v - \frac{\tilde{\Lambda}+ra/2}{\tilde{\Lambda}+a/2}\tilde{T}^*_v\tilde{T}_u = \frac{\tilde{\Lambda}+ra/2}{\tilde{\Lambda}+a/2}[\tilde{T}_u,\tilde{T}^*_v] + \left[\frac{\tilde{\Lambda}+ra/2-1}{\tilde{\Lambda}+a/2-1} - \frac{\tilde{\Lambda}+ra/2}{\tilde{\Lambda}+a/2}\right]\tilde{T}_u\tilde{T}^*_v. \end{aligned}$$

Therefore

$$\begin{aligned} U[S_u,S_v^*]U^* - [\tilde{T}_u,\tilde{T}^*_v] &= \left(\frac{\tilde{\Lambda}+ra/2}{\tilde{\Lambda}+a/2} - 1\right)[\tilde{T}_u,\tilde{T}^*_v] + \left(\frac{\tilde{\Lambda}+ra/2-1}{\tilde{\Lambda}+a/2-1} - \frac{\tilde{\Lambda}+ra/2}{\tilde{\Lambda}+a/2}\right)\tilde{T}_u\tilde{T}^*_v \\ &= \frac{(r-1)a/2}{\tilde{\Lambda}+a/2}[\tilde{T}_u,\tilde{T}^*_v] + \frac{(r-1)a/2}{(\tilde{\Lambda}+a/2-1)(\tilde{\Lambda}+a/2)}\tilde{T}_u\tilde{T}^*_v \in \mathcal{L}^{n/2,\infty} \end{aligned}$$

since $[\tilde{T}_u,\tilde{T}^*_v]$ (cf. [11]) and $(\tilde{\Lambda}+a/2)^{-1}$ belong to $\mathcal{L}^{n,\infty}$. $\square$

To consider general symbols, we need the following algebraic lemma.

Lemma 4.7. *Suppose that the given operators $A_i, \tilde{A}_j$ satisfy that $A_i - \tilde{A}_i$, $[A_i,A_j]$, $[\tilde{A}_i,\tilde{A}_j] \in \mathcal{L}^{n,\infty}$ and $[A_i,A_j] - [\tilde{A}_i,\tilde{A}_j] \in \mathcal{L}^{n/2,\infty}$ for $1 \le i,j \le 4$. Then*

$$[A_1A_2,A_3A_4] - [\tilde{A}_1\tilde{A}_2,\tilde{A}_3\tilde{A}_4] \in \mathcal{L}^{n/2,\infty}.$$

Corollary 4.8. *For polynomials p,q,ϕ,ψ, we have*

$$U[S^*_pS_q,S^*_\phi S_\psi]U^* - [\tilde{T}_{\overline{p}q},\tilde{T}_{\overline{\phi}\psi}] \in \mathcal{L}^{n/2,\infty}.$$

Proof. Apply Lemma (4.7) and Proposition (4.6). $\square$

Every $f \in \mathcal{C}^\infty(S)$ has a **Poisson integral extension** $\hat{f} \in \mathcal{C}^\infty(D)$, which is harmonic in the sense that it is annihilated by the so-called Hua operators [17, 21]. For any non-zero tripotent $c \in S_k$ there exists a continuous extension, again denoted by $\hat{f}$, onto the boundary component $c+D_c$. This extension is given by

$$\hat{f}(c+\zeta) = f_c^\wedge(\zeta)$$

for all $\zeta \in D_c^0$, where $f_c^\wedge$ denotes the Poisson extension, relative to the Shilov boundary S_c of D_c, for the restricted smooth function

$$f_c(\zeta) := f(c+\zeta), \qquad \zeta \in S_c.$$

Setting $\zeta = 0$ the Poisson extension $\hat{f}$ is well-defined on S_k.

Lemma 4.9. *For all $c \in S_k$ we have*

$$\widehat{\overline{p}q}(c) = (p_c|q_c)_{S_c}.$$

Proof. Let $h(z) = \widehat{\overline{p}q}(z)$ be the Poisson extension of $\overline{p(s)}q(s)$. For all $\zeta \in S_c$, we have $c + \zeta \in S$ and hence

$$h_c(\zeta) = h(c+\zeta) = \overline{p(c+\zeta)}q(c+\zeta) = \overline{p_c(\zeta)}q_c(\zeta)$$

Since $h_c(\zeta)$ is harmonic, the mean value property applied to the Peirce 0-space Z_c yields

$$h(c) = h_c(0) = h_c^\wedge(0) = \int_{S_c} d\zeta\, h_c(\zeta) = \int_{S_c} d\zeta\, \overline{p_c(\zeta)}q_c(\zeta) = (p_c|q_c)_{S_c}.$$

□

Proposition 4.10. *For polynomials $f \in \mathcal{P}(Z \times \overline{Z})$ we have $US_fU^* - \tilde{T}_{\hat{f}} \in \mathcal{L}^{n,\infty}$ and*

$$U[S_f^*, S_f]U^* - [\tilde{T}_{\hat{f}}^*, \tilde{T}_{\hat{f}}] \in \mathcal{L}^{n/2,\infty}.$$

Here $\hat{f}$ is the Poisson extension restricted to S_1.

Proof. Without loss of generality we may suppose that $f = \overline{p}q$ for $p, q \in \mathcal{P}(Z)$. By Theorem (3.1), Proposition (2.9) and Lemma (2.10), there exist $B \in \mathcal{B}_\Lambda$ and finitely many $p_i, q_i \in \mathcal{P}(Z)$ such that

$$S_f = PT_p^*T_qP = B + \sum_i S_{p_i}^* S_{q_i}.$$

By the definition of $\mathcal{B}_\Lambda$, Proposition (2.9) and Lemma (4.1), we have $S_f - \sum\limits_i S_{p_i}^* S_{q_i} \in \mathcal{L}^{n,\infty}$ and $[S_f^*, S_f] - \left[\sum\limits_i S_{q_i}^* S_{p_i}, \sum\limits_i S_{p_i}^* S_{q_i}\right] \in \mathcal{L}^{n/2,\infty}$. For any $c \in S_1$, the symbol map in [24, Theorem 3.12] is given by

$$(\sigma_1 S_f)(c) = (1_c \otimes 1_c)T_{p_c}^* T_{q_c}(1_c \otimes 1_c) = (p_c|q_c)_{S_c}(1_c \otimes 1_c),$$

and

$$\sigma_1\left(\sum_i S_{p_i}^* S_{q_i}\right)(c) = \sum_i \overline{p_i(c)}q_i(c)(1_c \otimes 1_c).$$

With Lemma (4.9) it follows that

$$\hat{f}|_{S_1} = \sum_i \overline{p}_i q_i.$$

Since $US_{p_i}^* S_{q_i} U^* = \tilde{B} + \tilde{T}_{p_i}^* \tilde{T}_{q_i} = \tilde{B} + \tilde{T}_{\overline{p}_i q_i}$, it follows that

$$US_fU^* = \tilde{T}_{\sum\limits_i \overline{p}_i q_i} + B = \tilde{T}_{\hat{f}} + B$$

for $B \in \mathcal{L}^{n,\infty}$. Therefore $US_fU^* - \tilde{T}_{\hat{f}} = US_fU^* - \sum\limits_i \tilde{T}_{\overline{p}_i q_i} \in \mathcal{L}^{n,\infty}$ and

$$\begin{aligned} & U\left[S_f^*, S_f\right]U^* - \left[\tilde{T}_{\hat{f}}^*, \tilde{T}_{\hat{f}}\right] \\ = \ & U\left(\left[S_f^*, S_f\right] - \left[\sum_i S_{q_i}^* S_{p_i}, \sum_i S_{p_i}^* S_{q_i}\right]\right)U^* - \left(U\left[\sum_i S_{q_i}^* S_{p_i}, \sum_i S_{p_i}^* S_{q_i}\right]U^* - \left[\tilde{T}_{\hat{f}}^*, \tilde{T}_{\hat{f}}\right]\right) \in \mathcal{L}^{n/2,\infty}. \end{aligned}$$

□

The explicit computation of the Dixmier trace uses the results of [11] on strictly pseudo-convex domains. Let $\Sigma = \{z \in Z : \mathrm{rank}(z) = 1\}$. Then $S_1 \subset \Sigma$ has the defining function $r(z) = (z|z) - 1$. Therefore the contact 1-form $\eta = (\partial r - \overline{\partial} r)/(2i)$ on S_1 [11, Section 2.1] is given by

$$\eta_c w = \frac{(w|c) - (c|w)}{2i}$$

for all $c \in S_1$ and $w \in T_c(\Sigma) \subset Z$. It follows that

$$(d\eta)_c(w_1, w_2) = \frac{(w_1|w_2) - (w_2|w_1)}{i}.$$

Since $T_c(\Sigma) = Z_c^2 \oplus Z_c^1 = \mathbf{C}\, c \oplus Z_c^1$ we may write $w = i\alpha\, c + v$, with $\alpha \in \mathbf{C}$ and $v \in Z_c^1$. Then $\eta_c(i\alpha\, c + v) = \alpha$. It follows that $Ker(\eta_c) = Z_c^1$ and the Reeb vector field $E_\perp$ [11, p. 614] is given by $c \mapsto ic$. Restricted to the tangent space $T_c(S_1) = i\mathbf{R}\, c \oplus Z_c^1$, the 2-form $d\eta$ has the radical $i\mathbf{R}\, c = \mathbf{R}\, E_\perp$ and is non-degenerate on $Ker(\eta_c)$. Every $\psi \in \mathcal{C}^\infty(S_1)$ defines a vector field $Z_\psi \in Ker(\eta)$ such that

$$d\eta(X, Z_\psi) = X\, \psi$$

for all vector fields $X \in Ker(\eta)$. For $\phi, \psi \in \mathcal{C}^\infty(S_1)$ we obtain the **boundary Poisson bracket**

$$\{\phi, \psi\}_\flat = d\eta(Z_\phi, Z_\psi) = Z_\phi\, \psi.$$

Theorem 4.11. *Let $f_j, g_j \in \mathcal{P}(Z \times \overline{Z})$. Then*

$$tr_\omega[S_{f_1}, S_{g_1}] \cdots [S_{f_n}, S_{g_n}] = C \int\limits_{S_1} ds \prod_{j=1}^n \{\hat{f}_j, \hat{g}_j\}_\flat,$$

where ds is the normalized K-invariant measure, $\hat{f}$ is the Poisson extension of f and $\{\phi, \psi\}_\flat$ denotes the boundary Poisson bracket. The constant

$$C = \frac{1}{(2\pi i)^n} \int\limits_{S_1} \eta \wedge \frac{(d\eta)^{n-1}}{n!}$$

will be computed in the following Proposition (4.12).

Proof. In general, if $T_1 \in \mathcal{L}^{n/2,\infty}$ and $T_2, \ldots, T_n \in \mathcal{L}^{n,\infty}$ then $T_1 T_2 \cdots T_n \in \mathcal{L}^1$ since $\mathcal{L}^{k,\infty} \subset \mathcal{L}^{k+\epsilon}$ for any $\epsilon > 0$. By Proposition (4.10) it follows that $U[S_{f_1}, S_{g_1}] \cdots [S_{f_n}, S_{g_n}]U^* - T \in \mathcal{L}^1$, where

$$T := [\tilde{T}_{\hat{f}_1}, \tilde{T}_{\hat{g}_1}] \cdots [\tilde{T}_{\hat{f}_n}, \tilde{T}_{\hat{g}_n}]$$

is a generalized Toeplitz operator on $H^2(S_1)$ of order $-n$. Applying [11, Theorem 3] it follows that

$$Tr_\omega[S_{f_1}, S_{g_1}] \cdots [S_{f_n}, S_{g_n}] = Tr_\omega(T) = \frac{1}{(2\pi)^n} \int\limits_{S_1} \eta \wedge \frac{(d\eta)^{n-1}}{n!}\, \sigma_{-n}(T)(x, \eta_x),$$

where η is the contact form. By [11, Section 4], T has the symbol

$$\sigma_{-n}(T) = \prod_{j=1}^n \sigma_{-1}[\tilde{T}_{\hat{f}_j}, \tilde{T}_{\hat{g}_j}] = \prod_{j=1}^n \frac{1}{i}\{\sigma_0 \tilde{T}_{\hat{f}_j}, \sigma_0 \tilde{T}_{\hat{g}_j}\}_\Sigma = \prod_{j=1}^n \frac{1}{i}\{\hat{f}_j^{(0)}, \hat{g}_j^{(0)}\}_\Sigma$$

in terms of the Poisson bracket of Σ. Here $\phi^{(0)}(t\, c) = \phi(c)$ denotes the 0-homogeneous extension of $\phi \in \mathcal{C}^\infty(S_1)$. Now the assertion follows, since by [11, Corollary 8] we have for $t = 1$

$$\frac{1}{i}\{\phi^{(0)}, \psi^{(0)}\}_\Sigma = Z_\phi \psi = \{\phi, \psi\}_\flat.$$

□

Let $V = Z_{e_1}^1$ be the Peirce 1-space for the minimal tripotent e_1. If $a \neq 2$ or $r = 1$, then V is an irreducible hermitian Jordan triple. If $a = 2$ and $r > 1$ then $Z = \mathbf{C}^{r \times (r+b)}$ and $V = \mathbf{C}^{(r-1) \times 1} \oplus \mathbf{C}^{1 \times (r+b-1)}$ is a direct sum of two hermitian Jordan triples of rank 1. For any irreducible hermitian Jordan triple V let Γ_V denote the Gindikin Γ-function for the radial cone $\Omega_V \subset V$ [14]. Let r_V, d_V, p_V denote the rank, dimension and genus of V, resp.

Proposition 4.12. *If $a \neq 2$ or $r = 1$, we have*

$$\frac{1}{(2\pi)^n}\int\limits_{S_1} \eta \wedge \frac{(d\eta)^{n-1}}{(n-1)!} = \frac{\Gamma_V(p_V - \frac{n-1}{r_V})}{\Gamma_V(p_V)};$$

If $a = 2$ we have

$$\frac{1}{(2\pi)^n}\int\limits_{S_1} \eta \wedge \frac{(d\eta)^{n-1}}{(n-1)!} = \frac{1}{\Gamma(r)\,\Gamma(r+b)}.$$

In the rank $r = 1$ case, where $Z = \mathbf{C}^d$ and $S_1 = \mathbf{S}^{2n-1}$, we have $n = d = 1 + b$ and both formulas imply

$$\frac{1}{(2\pi)^n}\int\limits_{S_1} \eta \wedge (d\eta)^{n-1} = 1.$$

Proof. Any irreducible hermitian Jordan triple V has a 'quasi-determinant' function $\Delta_V(u, v)$ such that the invariant measure on its conformal compactification M, containing V as an open dense subset of full measure, is a multiple of $\Delta_V(v, -v)^{-p_V}\, d\lambda(v)$, where $d\lambda(v)$ is Lebesgue measure for the normalized inner product. Moreover, by [13] we have the polar integration formula

$$\int\limits_V \frac{d\lambda(z)}{\pi^{d_V}} \Delta(z, -z)^{-p_V} = \frac{\Gamma_V(p_V - \frac{d_V}{r_V})}{\Gamma_V(p_V)}. \tag{4.3}$$

Let M denote the compact complex manifold of all Peirce 2-spaces $U \subset Z$ of rank 1. There is a canonical map

$$\pi : \Sigma \to M$$

which maps $z \in \Sigma$ onto its Peirce 2-space Z_z^2. In this way, Σ becomes a hermitian holomorphic line bundle over M which can be identified with the tautological line bundle $\mathcal{L} = \bigcup_{U \in M} U$. The subset $S_1 \subset \Sigma$ corresponds to the circle bundle $\bigcup_{U \in M} S_U$, where $S_U \approx \mathbf{S}^1$ is the Shilov boundary of $U \in M$. The holomorphic map π satisfies

$$\ker(d_c\pi) = Z_c^2,$$

since $Z_c^2 \subset \ker(d_c\pi)$ and both spaces are 1-dimensional. Therefore $d\eta$ vanishes on

$$T_c(S_1) \cap \ker(d_c\pi) = i\mathbf{R} \cdot c.$$

As a consequence there exists a K-invariant 2-form Θ on M such that $d\eta = \pi^*\Theta$. Now η, restricted to S_U, is the usual contact form on $\mathbf{S}^1$ of volume 2π. It follows that

$$\frac{1}{(2\pi)^n}\int\limits_{S_1} \eta \wedge \frac{(d\eta)^{n-1}}{(n-1)!} = \frac{1}{(2\pi)^n}\int\limits_{S_1} \eta \wedge \frac{\pi^*\Theta^{n-1}}{(n-1)!} = \frac{1}{(2\pi)^{n-1}}\int\limits_M \frac{\Theta^{n-1}}{(n-1)!}.$$

In order to compute this integral, let $V = Z_{e_1}^1$. A local coordinate for M is given by the map $\sigma := \pi \circ \tau : V \to M$, where $\tau : V \to \Sigma$ is defined by

$$\tau(v) = e_1 + v + \{ve_1^*v\}.$$

The semi-simple part K' of K acts transitively on M, and induces a 'Moebius-type' biholomorphic action on V such that σ becomes K'-equivariant. We have

$$\tau^* d\eta = \tau^*(\pi^*\Theta) = \sigma^*\Theta.$$

Since $(d_0\tau)v = v$ at the origin $0 \in V$, the pull-back $\tau^*(d\eta)|_v(v_1, v_2) = (d\eta)_{\tau(v)}((d_v\tau)v_1, (d_v\tau)v_2)$ satisfies

$$\sigma^*\Theta|_0(v_1, v_2) = \tau^*(d\eta)|_0(v_1, v_2) = \frac{(v_1|v_2) - (v_2|v_1)}{i}.$$

Using complex coordinates v_j with respect to an orthonormal basis of $V = T_0(V)$ this means

$$\sigma^*\Theta|_0 = \sum_{j=1}^{n-1} \frac{d\overline{v}_j \wedge dv_j}{i}.$$

Now assume that $a \neq 2$ or $r = 1$. Then M is irreducible. Since Θ is invariant under K, it follows that $\sigma^*\Theta$ is invariant under the Moebius action. Since $d_V = n-1$, we obtain for the volume form

$$\frac{\sigma^*\Theta^{n-1}}{(n-1)!} = C \cdot \Delta_V(v,-v)^{-p_V}\, d\lambda(v),$$

where C is a constant. Evaluating at $0 \in V$ and using

$$d\lambda(v) = \prod_{j=1}^{n-1} \frac{d\overline{v}_j \wedge dv_j}{2i} = \frac{1}{(n-1)!}\left(\sum_{j=1}^{n-1} \frac{d\overline{v}_j \wedge dv_j}{2i}\right)^{n-1} = \frac{1}{2^{n-1}} \frac{(\sigma^*\Theta|_0)^{n-1}}{(n-1)!}. \tag{4.4}$$

it follows that $C = 2^{n-1}$. Since σ is a Zariski dense open embedding of full measure we obtain

$$\frac{1}{(2\pi)^{n-1}} \int\limits_M \frac{\Theta^{n-1}}{(n-1)!} = \frac{1}{(2\pi)^{n-1}} \int\limits_V \frac{\sigma^*\Theta^{n-1}}{(n-1)!} = \frac{C}{2^{n-1}} \int\limits_V \frac{d\lambda(v)}{\pi^{n-1}}\, \Delta_V(v,-v)^{-p_V} = \frac{\Gamma_V(p_V - \frac{n-1}{r_V})}{\Gamma_V(p_V)}$$

by applying (4.3) to the irreducible hermitian Jordan triple $V = Z^1_{e_1}$. Now assume $a = 2$ and $r > 1$. Then $Z = \mathbf{C}^{r\times(r+b)}$ and M is reducible. More precisely,

$$\Sigma = \{z \in \mathbf{C}^{r\times(r+b)} : \, rank(z) = 1\} = \{\xi_1\xi_2 : \, 0 \neq \xi_1 \in \mathbf{C}^{r\times 1},\, 0 \neq \xi_2 \in \mathbf{C}^{1\times(r+b)}\}.$$

Consider the associated projective spaces $M_1 = \mathbf{P}(\mathbf{C}^{r\times 1}) = \{[\xi_1] : \, 0 \neq \xi_1 \in \mathbf{C}^{r\times 1}\} \approx \mathbf{P}^{r-1}$ and $M_2 = \mathbf{P}(\mathbf{C}^{1\times(r+b)}) = \{[\xi_2] : \, 0 \neq \xi_2 \in \mathbf{C}^{1\times(r+b)}\} \approx \mathbf{P}^{r+b-1}$. Then $M = M_1 \times M_2$ is a direct product via the identification $([\xi_1],[\xi_2]) \mapsto Ran(\xi_1\xi_2)$. For $i \in \{1,2\}$, the map $\pi_i : \Sigma \to M_i$ given by $\xi_1\xi_2 \mapsto [\xi_i]$ is well-defined and the canonical map $\pi : \Sigma \to M_1 \times M_2$ is a product

$$\pi(\xi_1\xi_2) = ([\xi_1],[\xi_2]) = (\pi_1(\xi_1\xi_2), \pi_2(\xi_1\xi_2)).$$

Now $\Theta = \Theta_1 \oplus \Theta_2$ is the direct sum of K'-invariant 2-forms Θ_i on M_i. Using the binomial theorem for (commuting) 2-forms, the corresponding volume form is

$$\frac{\Theta^{n-1}}{(n-1)!} = \frac{\Theta_1^{n_1}}{n_1!} \wedge \frac{\Theta_2^{n_2}}{n_2!}$$

for the dimensions $n_1 = r-1, n_2 = r+b-1$ adding up to $n_1 + n_2 = 2(r-1)+b = n-1$. It follows that

$$\frac{1}{(2\pi)^{n-1}} \int\limits_M \frac{\Theta^{n-1}}{(n-1)!} = \frac{1}{(2\pi)^{n_1}} \int\limits_{M_1} \frac{\Theta_1^{n_1}}{n_1!} \frac{1}{(2\pi)^{n_2}} \int\limits_{M_2} \frac{\Theta_2^{n_2}}{n_2!}.$$

In order to compute these integrals, put $V_1 := \mathbf{C}^{(r-1)\times 1}$ and $V_2 = \mathbf{C}^{1\times(r+b-1)}$. Then

$$V = Z^1_{e_1} = \{\begin{pmatrix} 0 & v_2 \\ v_1 & 0 \end{pmatrix} : \, v_i \in V_i\} \approx V_1 \times V_2.$$

The local coordinate $\sigma(v_1,v_2) = (\sigma_1(v_1), \sigma_2(v_2))$ is of product type, where $\sigma_i(v_i) := [1, v_i]$. In fact, putting $v = \begin{pmatrix} 0 & v_2 \\ v_1 & 0 \end{pmatrix} \in V$, we obtain

$$\tau(v) = e_1 + v + \{ve_1^*v\} = \begin{pmatrix} 1 & v_2 \\ v_1 & v_1v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ v_1 \end{pmatrix} \begin{pmatrix} 1 & v_2 \end{pmatrix}$$

and hence

$$\sigma(v) = \pi(\tau(v)) = [\begin{pmatrix} 1 \\ v_1 \end{pmatrix}], [\begin{pmatrix} 1 & v_2 \end{pmatrix}].$$

The semi-simple part $K' = SU(r) \times SU(r+b)$ of $K = S(U(r) \times U(r+b))$ acts transitively on each factor M_i and induces a 'Moebius-type' biholomorphic action on V_i such that σ_i becomes K'-equivariant. Since Θ_i is invariant under K', it follows that $\sigma_i^*\Theta_i$ is invariant under this Moebius action. This implies for the volume form

$$\frac{\sigma_i^*\Theta_i^{n_i}}{n_i!} = C_i \cdot (1 + (v_i|v_i))^{-1-n_i}\, d\lambda_i(v_i),$$

where C_i is a constant. Using the relation

$$d\lambda_i(v_i) = \frac{1}{2^{n_i}} \frac{(\sigma_i^* \Theta_i|_0)^{n_i}}{n_i!}$$

analogous to (4.4), it follows that $C_i = 2^{n_i}$. Since $\sigma_i : V_i \to M_i$ is a Zariski dense open embedding of full measure we obtain

$$\frac{1}{(2\pi)^{n_i}} \int\limits_{M_i} \frac{\Theta_i^{n_i}}{n_i!} = \frac{1}{(2\pi)^{n_i}} \int\limits_{V_i} \frac{\sigma_i^* \Theta_i^{n_i}}{n_i!} = \frac{C_i}{2^{n_i}} \int\limits_{V_i} \frac{d\lambda_i(v_i)}{\pi^{n_i}} (1 + (v_i|v_i)^{-1-n_i} = \frac{\Gamma(1)}{\Gamma(1+n_i)} = \frac{1}{n_i!}$$

by applying (4.3) to the irreducible hermitian Jordan triple V_i. □

Finally, let us mention a relation involving numerical invariants of the domain D and $V = Z^1_{e_1}$, which would make the formulas more tractable.

Lemma 4.13. *Suppose that $a \neq 2$. Then the rank r_V and the genus p_V of $V = Z^1_{e_1}$ satisfy the relation*

$$r_V \, p_V = ra + b.$$

As a consequence, the Γ-function quotient in Proposition (4.12) *can also be expressed as*

$$\frac{\Gamma_V(p_V - \frac{n-1}{r_V})}{\Gamma_V(p_V)} = \frac{\Gamma_V(\frac{a}{r_V})}{\Gamma_V(\frac{ra+b}{r_V})}.$$

Proof. We use the classification of hermitan Jordan triples [19, 20]. For $a = 2$, we obtain the Jordan triples $Z = \mathbf{C}^{r \times (r+b)}$ of type (I), for which the relation does not hold. The other cases are listed in the following table

type	Z	rank	a	b	V	r_V	p_V
(II)	$\mathbf{C}^{(2r+\epsilon)\times(2r+\epsilon)}_{\text{asym}}$	r	4	2ϵ	$\mathbf{C}^{2\times(2(r-1)+\epsilon)}$	2	$2r+\epsilon$
(III)	$\mathbf{C}^{r\times r}_{\text{symm}}$	r	1	0	$\mathbf{C}^{r-1}$	1	r
(IV)	$\mathbf{C}^{d}_{\text{spin}}$	2	d-2	0	$\mathbf{C}^{d-2}_{\text{spin}}$	2	d-2
(V)	$\mathbf{O}^{1\times 2}_{\mathbf{C}}$	2	6	4	$\mathbf{C}^{5\times 5}_{\text{asym}}$	2	8
(VI)	$\mathcal{H}_3(\mathbf{O})\otimes\mathbf{C}$	3	8	0	$\mathbf{O}^{1\times 2}_{\mathbf{C}}$	2	12

□

References

[1] J. Arazy, S. Fisher, S. Janson and J. Peetre, *An identity for reproducing kernels in a planar domain and Hilbert-Schmidt Hankel operators,* J. reine angew. Math. 406 (1990), 179–199.

[2] J. Arazy and H. Upmeier, *Boundary measures for symmetric domains and integral formulas for the discrete Wallach points*, Int. Eq. Op. Th. 47 (2003), 375-434.

[3] C. Berger and L. Coburn, *Wiener-Hopf operators on U_2*, Int. Eq. Op. Th. 2 (1979), 139-173.

[4] C. Berger, L. Coburn and A. Korányi, *Opérateurs de Wiener-Hopf sur les sphères de Lie,* C.R. Acad. Sci. Paris 290 (1980), 989-991.

[5] A. Connes, *The action functional in noncommutative geometry*, Comm. Math. Phys. 117 (1988), 673–683.

[6] A. Connes, *Noncommutative geometry*, Academic Press, San Diego, 1994.

[7] J. Dixmier, *C^*-Algebras,* North-Holland, Amsterdam, 1977.

[8] R. Douglas, X. Tang and G. Yu, *An analytic Grothendieck-Riemann-Roch theorem,* preprint, arXiv:1404.4396 (2014).

[9] M. Englis and J. Eschmeier, *Geometric Arveson-Douglas conjecture,* Adv. Math. 9(2015), 606–630.

[10] M. Englis, K. Guo and G. Zhang, *Toeplitz and Hankel operators and Dixmier traces on the unit ball of $\mathbb{C}^n$*, Proc. Amer. Math. Soc. 137 (2009), 3669–3678.

[11] M. Englis and G. Zhang, *Hankel operators and the Dixmier trace on strictly pseudoconvex domains*, Documenta Math. 15 (2010),601–622.

[12] M. Englis and R. Rochberg, *The Dixmier trace of Hankel operators on the Bergman space,* J. Funct. Anal. 257 (2009), 1445–1479.

[13] M. Englis and H. Upmeier, *Real Berezin transform and asymptotic expansion for symmetric spaces of compact and non-compact type*, Operator Theory, Advances and Applications 220, Birkhäuser (2010), 97-114.

[14] J. Faraut and A. Korányi, *Analysis on Symmetric Cones*, Clarendon Press, Oxford, 1994.

[15] K. Guo and K. Wang, *Essentially normal Hilbert modules and K-homology,* Math. Ann. 340 (2008), 907–934.

[16] K. Guo, K. Wang and G. Zhang, *Trace formulas and p-essentially normal properties of quotient modules on the bidisk*, J. Operator Th. 67 (2012), 511-535.

[17] A. Korányi, *Poisson integrals and boundary components of symmetric spaces*, Invent. Math. 34 (1976), 19–35.

[18] W.J. Helton and R. Howe, *Traces of commutators of integral operators,* Acta Math. 135 (1975), 271–305.

[19] O. Loos, *Jordan Pairs,* Lecture Notes in Math. 460, Springer (1975).

[20] E. Neher, *Jordan Triple Systems by the Grid Approach*, Lect. Notes in Math. 1280, Springer (1987)

[21] H. Schlichtkrull, *On the boundary behaviour of generalized Poisson integrals on symmetric spaces*, Trans. Amer. Math. Soc. 290 (1985), 273-280.

[22] R. Stanley, *Some combinatorial properties of Jack symmetric functions*, Adv. Math. 77 (1989), 76–115.

[23] H. Upmeier, *Toeplitz operators on bounded symmetric domains*, Trans. Amer. Math. Soc. 280 (1983), 221-237.

[24] H. Upmeier, *Toeplitz C^*-algebras on bounded symmetric domains*, Ann. Math. 119 (1984), 549-576.

[25] H. Upmeier, *Jordan algebras and harmonic analysis on symmetric spaces*, Amer. J. Math. 108 (1986), 1-25.

[26] H. Upmeier, *Multivariable Toeplitz operators and index theory*, Progress in Math. 84, Birkhäuser (1991), 275-288.

[27] H. Upmeier, *Index theory for multivariable Wiener-Hopf operators*, J. Reine Angew. Math. 384 (1988), 57-79.

[28] M. Wodzicki, *Local invariants of spectral asymmetry*, Invent. Math. 75 (1984), 143-178.

Fachbereich Mathematik, Universität Marburg, Marburg, 35032, Germany

E-mail address: upmeier@mathematik.uni-marburg.de

School of Mathematical Sciences, Fudan University, Shanghai, 200433, P. R. China

E-mail address: kwang@fudan.edu.cn